CHAPTER 3

Part 1. Some properties of Γ.

Throughout Part 1 of this chapter, we assume that the quotient G/Γ is compact. Our main purpose is to prove the following results (i)~(iv) (particularly (iv)).

(i) The commutator subgroup $[\Gamma, \Gamma]$ of Γ is of finite index in Γ. Moreover, if Γ is torsion-free, then the index $(\Gamma : [\Gamma, \Gamma])$ is a divisor of $P(1)^2$, where

$$P(u) = \prod_{i=1}^{g}(1 - \pi_i u)(1 - \pi_i' u)$$

is the numerator of the main factor of $\zeta_\Gamma(u)$ (Theorem 2, §6).

(ii) $\Gamma_{\mathbf{R}}$ has no non-trivial deformation in $G_{\mathbf{C}} = PL_2(\mathbf{C})$ (Theorem 3, §7).

(iii) Γ is residually finite. Moreover, Γ contains a torsion-free subgroup of finite index (Theorem 4, §9).

(iv) The field $F = \mathbf{Q}((\operatorname{tr}\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$ is an algebraic number field. Moreover, there is a quaternion algebra A over F, which is uniquely determined by Γ, such that for any field $K \subset \mathbf{C}$ the following two statements (a), (b) are equivalent:
  (a) There is an element $t \in G_{\mathbf{C}} = PL_2(\mathbf{C})$ such that $t^{-1}\Gamma_{\mathbf{R}} t \subset PL_2(K)$.
  (b) K contains F and $A \otimes_F K \cong M_2(K)$.
  Furthermore, $\Gamma_{\mathbf{R}}$ can be considered as a subgroup of $A^\times/F^\times$ (Theorem 5, Proposition 6;§12, §13).

We begin with some preliminaries; then we shall prove Theorem 1 (§5) which asserts $H^1(\Gamma_{\mathbf{R}}, \rho_n) = \{0\}$ $(n \geq 0)$, where ρ_n is the symmetric tensor representation of $G_{\mathbf{R}}$ of degree $2n$ (see §3). This is a consequence of Eichler-Shimura's isomorphism (see §4), Kuga's lemma (Lemma 10 of Chapter 1), and our remarks on cohomology groups (§1 §2). Now, Theorem 1 is basic for all our results (i)-(iv). In fact, (i) and (ii) are almost direct consequences of $H^1(\Gamma_{\mathbf{R}}, \rho_n) = \{0\}$ for $n = 0$ and $n = 1$ respectively; and (iii), (iv) are results of our study of "deformation variety" of $\Gamma_{\mathbf{R}}$ in $G_{\mathbf{C}}$, of which (ii) is the starting point.

Finally, we remark that some of our results are valid also for more general dense subgroups $\Gamma_{\mathbf{R}}$ of $G_{\mathbf{R}}$ satisfying some conditions (see Remark 1 in §7).