Part 3B. Unique existence of an invariant S-operator on "arithmetic" algebraic function fields (including G_p-fields) over any field of characteristic zero.

Unique existence of invariant S-operator on ample (arithmetic) L/k.

§45.

[1]. In §41 (Part 3A), we considered the algebraic function fields $L/\mathbf{C}$ satisfying (L1), (L2), and proved Theorem 9 for such fields. In particular, we proved that if L is ample, then there exists a unique $\mathrm{Aut}_{\mathbf{C}} L$-invariant S-operator on L. Our purpose here is to generalize this result to the cases where the constant field k of L is an arbitrary field of characteristic zero (instead of $\mathbf{C}$). First, we must define the fields L/k. This is completely parallel to the definition of $L/\mathbf{C}$ (§41); namely, our object will be the following field L/k:

Definition. k is any field of characteristic 0, and L is any one-dimensional extension of k not assumed to be finitely generated over k, but assumed to satisfy:

$(L0)_k$ k is algebraically closed in L;

$(L1)_k$ Let $\mathcal{L}_0$ be the set of all finitely generated extensions L_0/k contained in L such that L/L_0 is normally algebraic. Then $\mathcal{L}_0$ is non-empty;

$(L2)_k$ For each $L_0 \in \mathcal{L}_0$ and a prime divisor P_0 of L_0/k, denote by $e_0(P_0)$ the ramification index of P_0 in L/L_0. Then $e_0(P_0) = 1$ for almost all P_0, and the quantity

$$V(L_0) = 2g_0 - 2 + \sum_{P_0}\left(1 - \frac{1}{e_0(P_0)}\right)\deg P_0 \tag{128}$$

is positive, where g_0 is the genus of L_0/k.

Remark 1. Remark 1 of §41 is also valid here.

Remark 2. If $k = \mathbf{C}$, this coincides with the definition of $L/\mathbf{C}$ of §41.

[2]. The arguments of [2] [3] of §41 are also applicable to this general case; so, all definitions and results of [2] [3] §41 are directly carried over to this case if we only replace $\mathbf{C}$ by k. In particular, $\mathcal{L}_0$ always contains a minimal element (with respect to $\subset$), and L is called *simple* if it is unique, and *ample* (or *arithmetic*) if it is not unique. Moreover, L is ample if and only if $\mathrm{Aut}_k L$ is non-compact. The definitions of $D(L)$ and $d : L \to D(L)$ are also exactly parallel to the case of $k = \mathbf{C}$ ([4] §41).

Remark 3. There is one point where we need a slight modification of our argument: In [3] §41, we used the finiteness of $\mathrm{Aut}\{L_0, e_0\}$ (to prove Proposition 14), and reduced this finiteness proof to the well-known finiteness of $N(\Delta)/\Delta$, where Δ is the fuchsian group corresponding to $\{L_0, e_0\}$, and $N(\Delta)$ is its normalizer in $G_{\mathbf{R}}$. For the general case, the finiteness of $\mathrm{Aut}\{L_0, e_0\}$ is proved in the following way: First, if the genus g_0 of L_0 is