Part 2. Full $G_{\mathfrak{p}}$-subfields over algebraic number fields.

The readers are suggested to recall the definitions of full $G_{\mathfrak{p}}$-subfield (§4) and quasi-irreducibility (§16) of a $G_{\mathfrak{p}}$-field over $\mathbf{C}$. Throughout the following, an algebraic number field always means a *finite* algebraic extension of the field of rational numbers $\mathbf{Q}$.

Main results.

§18. Our main purpose in Part 2 of this chapter is to prove the following two theorems, Theorem 4 and Theorem 5. Later, we shall give some supplementary results (see §32 ~ §36).

THEOREM 4. *Every $G_{\mathfrak{p}}$-field over $\mathbf{C}$ contains a full $G_{\mathfrak{p}}$-subfield over an algebraic number field.*

If we impose quasi-irreducibility condition on a $G_{\mathfrak{p}}$-field over $\mathbf{C}$, then we get an essentially stronger result, as follows.

THEOREM 5. *Every quasi-irreducible $G_{\mathfrak{p}}$-field L over $\mathbf{C}$ contains a unique full $G_{\mathfrak{p}}$-subfield L_{k_0} over an algebraic number field k_0 satisfying the following properties; namely, if k is any subfield of $\mathbf{C}$, then there is a full $G_{\mathfrak{p}}$-subfield L_k over k if and only if k contains k_0, and moreover if k is such a field, then L_k is unique and is given by $L_k = L_{k_0} \cdot k$.*

In short, every quasi-irreducible $G_{\mathfrak{p}}$-field over $\mathbf{C}$ contains a smallest full $G_{\mathfrak{p}}$-subfield over an algebraic number field, and all other full $G_{\mathfrak{p}}$-subfields are its constant field extensions. This will be referred to as *the existence and essential uniqueness of a full $G_{\mathfrak{p}}$-subfield over an algebraic number field* of a quasi-irreducible $G_{\mathfrak{p}}$-field over $\mathbf{C}$. Some variations of Theorem 5 will be given in §32, §33.

Although Theorem 5 is essentially stronger (and hence more noteworthy) than Theorem 4, it is almost a formal consequence of Theorem 4. Thus, our first task is to show this.

Reducing Theorem 5 to Theorem 4.

§19. In general, if $L \supset K_1, K_2$ are overfields of a field k such that $L = K_1K_2$ and that K_1, K_2 are linearly disjoint over k, and if σ_1, σ_2 are automorphisms of K_1, K_2 respectively such that $\sigma_1|k = \sigma_2|k$, then there is a unique automorphism of L whose restrictions to K_1, K_2 coincide with σ_1, σ_2 respectively. This automorphism of L will be denoted by $\sigma_1 \otimes \sigma_2$. The identity automorphism of a field K will be denoted by 1_K.