Part 2.[10] Detailed study of elements of Γ with parabolic and elliptic real parts; the general formula for $\zeta_\Gamma(u)$.

Let Γ be a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}} = PSL_2(\mathbf{R}) \times PSL_2(k_{\mathfrak{p}})$ with finite volume quotient G/Γ and with dense image of projection in each component of G. In the previous part of this chapter, we defined the ζ-function

$$\zeta_\Gamma(u) = \prod_{P \in \wp(\Gamma)} (1 - u^{\deg P})^{-1}$$

for such a group Γ (§6) and carried out its computation under the two assumptions: (a) G/Γ is compact, (b) Γ is torsion-free. (See Theorems 1, 2).

In the following Part 2, *we shall drop the above two assumptions* (a), (b), and after studying in detail the elements of Γ with parabolic real parts (§25 ~ §28, Theorem 3) and those with elliptic real parts (including in particular the torsion elements of Γ; §29~ §34, Theorems 4 ~ 6), we shall proceed to *prove a general formula for* $\zeta_\Gamma(u)$ by generalizing the previous computations (§35 ~ §38, Theorem 7). The main results are as follows:

1. Let $\gamma \in \Gamma$ be such that $\gamma_{\mathbf{R}}$ is parabolic.[11] Let H^0 be the centralizer of γ and let H be the normalizer of H^0 (both considered in Γ). Then (i) $k_{\mathfrak{p}} = \mathbf{Q}_p$ holds, (ii) H is conjugate in $G_{\mathbf{R}} \times PL_2(\mathbf{Z}_p)$ to the group

$$B^{(d)} = \left\{ \begin{pmatrix} p^{dk} & b \\ 0 & p^{-dk} \end{pmatrix} \middle|\, k \in \mathbf{Z}, b \in \mathbf{Z}^{(p)} \right\} \tag{102}$$

(where d is a positive integer well-defined by H), and by this, H^0 corresponds to the subgroup $\begin{pmatrix} 1 & \mathbf{Z}^{(p)} \\ 0 & 1 \end{pmatrix}$ of $B^{(d)}$ (Theorem 3, §25). By this theorem we can derive everything we need about such elements γ.

2. Let $\gamma \in \Gamma$ be such that $\gamma_{\mathbf{R}}$ is elliptic.[12] Put $\Gamma^0 = \Gamma \cap (G_{\mathbf{R}} \times V)$ with $V = PSL_2(O_{\mathfrak{p}})$, and for each $l \geq 0$ put $T^l = \Gamma \cap \left\{ G_{\mathbf{R}} \times V \begin{pmatrix} \pi^l & 0 \\ 0 & \pi^{-l} \end{pmatrix} V \right\}$, π being a prime element of $k_{\mathfrak{p}}$. Then our results here are the following:

(i) we parametrize the set of all Γ^0-conjugacy classes contained in $\{\gamma\}_\Gamma$ in a nice way as, say,

$$\{\gamma\}_\Gamma = \bigcup_{k,\,\mu} \{\gamma_{k\mu}\}_{\Gamma^0}; \quad k = 0, 1, 2, \cdots; \quad \mu = 1, \cdots, n_k;$$

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[10] The author regrets that, despite his promise, he has failed to give a computation of L-functions $L_\Gamma(u, \chi)$ here. The reason is that when χ is not a real character, his definition of $L_\Gamma(u, \chi)$ was not adequate, and it still remains for him to find its best definition.

[11] An element $x \in G_{\mathbf{R}}$ is called parabolic if its eigenvalues are $\pm\{1,\ 1\}$ and $x \neq 1$.

[12] An element $x \in G_{\mathbf{R}}$ is called elliptic if its eigenvalues are imaginary.