CHAPTER 1

Part 1. The group Γ and its ζ-function.

In Part 1 of this chapter, we shall define the ζ-function

$$\zeta_\Gamma(u) = \prod_P (1 - u^{\deg P})^{-1}$$

of Γ, and prove that

$$(20) \qquad \zeta_\Gamma(u) = \frac{\prod_{i=1}^{g}(1-\pi_i u)(1-\pi_i' u)}{(1-u)(1-q^2 u)} \times (1-u)^{(q-1)(g-1)};$$
$$q = N\mathfrak{p},\ g \geq 2,\ \pi_i \pi_i' = q^2\ (1 \leq i \leq g)$$

holds, if G/Γ is compact and Γ is torsion-free. We shall also prove the inequality; $|\pi_i|$, $|\pi_i'| \leq q^2$, π_i, $\pi_i' \neq 1, q^2$, by applying Lemma 10 (M.Kuga), §21. These results, particularly the existence of the factor $(1-u)^{(q-1)(g-1)}$, give a starting point of our problems described in the introduction. Our formula (20) is, modulo some group theory of $PL_2(k_\mathfrak{p})$, a consequence of Eichler-Selberg trace formula for the Hecke operators in the space of certain automorphic forms of weight 2. However, the proof, starting at Eichler-Selberg formula and ending at (20), is by no means simple, mainly because we do not have a simple proof of Lemma 3 (§13).[1] Finally, we point out that there is also a difference in the standpoint; Eichler-Selberg's left side of the formula comes to the right side of ours; (20). For us, the subject is the set of "elliptic Γ-conjugacy classes", and not the Hecke operator.

We shall begin with the definition of the group Γ.

Discrete subgroup Γ.

§1. Let

$$(1) \qquad G = PSL_2(\mathbf{R}) \times PSL_2(k_\mathfrak{p})$$

be considered as a topological group, and for each subset S of G, we denote by $S_\mathbf{R}$ resp. $S_\mathfrak{p}$ the set-theoretical projections of S to $\mathbf{R}$-component (i.e. the first component) resp.

[1] We can also prove (20) (for $\mathfrak{p} \nmid 2$) by using the spectral decomposition of $L^2(G/\Gamma)$.