6. Hyperbolic groups.

In this section we will explore some of the basic properties of hyperbolic groups. The notion of a hyperbolic group was introduced by Gromov around 1985. They arise in many different contexts, and there is a sense in which a "generic" finitely presented group is hyperbolic.

For much of the discussion we will just deal with geodesic spaces. One can get quite a long way with just elementary metric space theory as we shall see.

6.1. Definition of a hyperbolic space.

Let (X, d) be a geodesic metric space.

Definition : A *(geodesic) triangle*, T, in X consists of three geodesic segments, (α, β, γ) cyclically connecting three point (called the *vertices* of T). We refer to the geodesics segments as the *sides* of T.

Definition : If $k \geq 0$, a point, $p \in X$ is said to be a *k-centre* for the triangle T if $\max\{d(p, \alpha), d(p, \beta), d(p, \gamma)\} \leq k$.

See Figure 6a. (In the figures in this section, geodesics are often depicted curved inwards, rather than as euclidean straight lines. This is meant to evoke the Poincaré model of the hyperbolic plane, to which hypebolic spaces have a more natural resemblence.)

Definition : We say that X is *k-hyperbolic* if every triangle has a k-centre.

Definition : We say that X is *hyperbolic* if it is k-hyperbolic for some $k \geq 0$. We refer to such a k as a *hyperbolicity constant* for X.

Examples.

(1) Any space of finite diameter, k, is k-hyperbolic.

(2) Any tree is 0-hyperbolic

(3) The hyperbolic plane $\mathbf{H}^2$ is $(\frac{1}{2} \log 3)$-hyperbolic.