5. Hyperbolic geometry.

Here we give a brief outline of some of the main features of hyperbolic geometry. Again, this will serve mainly as a source of examples and motivation, and we will not give detailed proofs. One-dimensional hyperbolic space is just the real line, so we begin in dimension 2. The main ideas generalise to higher dimensions.

5.1. The hyperbolic plane.

We describe the "Poincaré model" for the hyperbolic plane. For this it is convenient to use complex coordinates. Let $D = \{z \in \mathbf{C} \mid |z| < 1\}$. Suppose $\alpha : I \longrightarrow D$ is a smooth path. We write $\alpha'(t) \in \mathbf{C}$ for the complex derivative at t. Thus, $|\alpha'(t)|$ is the "speed" at time t. The euclidean length of α is thus given by the formula $l_E(\alpha) = \int_I |\alpha'(t)|dt$. This is equal to the "rectifiable" length as defined in Section 3.

We now modify this by the introduction a scaling factor, $\lambda : D \longrightarrow (0, \infty)$. The appropriate formula is: $\lambda(z) = 2/(1 - |z|^2)$. The hyperbolic length of α is thus given by $l_H(\alpha) = \int_I \lambda(\alpha(t))|\alpha'(t)|dt$.

Note that as z approaches ∂D in the euclidean sense, then $\lambda(z) \rightarrow \infty$. Thus close to ∂D, things big in hyperbolic space may look very small to us in euclidean space. Indeed, since $\int_0^\infty \frac{2}{1-x^2}dx = \infty$, one needs to travel an infinite hyperbolic distance to approach ∂D. For this reason, ∂D is often referred to as the *ideal* boundary — we never actually get there.

Given $x, y \in D$, write $\rho(x, y) = \inf\{l_H(\alpha)\}$ as α varies over all smooth paths from x to y. In fact, the minimum is attained — there is always a smooth geodesic from x to y. The remark about the ideal boundary in the previous paragraph boils down to saying that this metric is complete. Moreover, if we want to get between two points x and y as quickly as possible, it would seem a good idea to move a little towards the centre of the disc, in the euclidean sense. Thus we would expect hyperbolic geodesics approach the middle of the disc relative to their euclidean counterparts.

For a more precise analysis, we need the notion of a *Möbius transformation*. This is a map $f : \mathbf{C} \cup \{\infty\} \longrightarrow \mathbf{C} \cup \{\infty\}$ of the form $f(z) = \frac{az+b}{cz+d}$ for constants $a, b, c, d \in \mathbf{C}$ with $ad - bc \neq 0$. We