3. Quasi-isometries.

In this section we will define the notion of a "quasi-isometry" — one of the fundamental notions in geometric group theory. First, though, we need to describe some more general terminology, and make a few technical observations. Many of the technical details mentioned below can be forgotten about in the main cases of interest to us, where the statements will be apparent. However, we might as well state them in general.

3.1. Metric Spaces.

Let (M, d) be a metric space.

Notation. Given $x \in M$ and $r \geq 0$, write $N(x, r) = \{y \in M \mid d(x, y) \leq r\}$ for the closed *r-neighbourhood* of x in M. If $Q \subseteq M$, write $N(Q, r) = \bigcup_{x \in Q} N(x, r)$. We say that Q is *r-dense* in M if $M = N(Q, r)$. We say that Q is *cobounded* if it is r-dense for some $r \geq 0$. Write $\operatorname{diam}(Q) = \sup\{d(x, y) \mid x, y \in Q\}$ for the *diameter* of Q. We say that Q is *bounded* if $\operatorname{diam}(Q) < \infty$. Note that any compact set is bounded.

Definition : Let $I \subseteq \mathbf{R}$ be an interval. A *(unit speed) geodesic* is a path $\gamma : I \longrightarrow M$ such that $d(\gamma(t), \gamma(u)) = |t - u|$ for all $t, u \in I$.

(Sometimes, we may talk about a *constant speed* geodesic, where $d(\gamma(t), \gamma(u)) = \lambda|t - u|$ for some constant "speed" $\lambda \geq 0$.)

Note that a geodesic is an arc, i.e. injective (unless it has zero speed).

Warning: This terminology differs slightly from that commonly used in riemannian geometry. There a "geodesic" is a path satisfying the geodesic equation. This is equivalent to being *locally* geodesic of consant speed in our sense.