3

DPN surfaces of elliptic type

3.1. Fundamental chambers of $W^{(2,4)}(S)$ for elliptic type

The most important property of lattices S of elliptic type is that the subgroup $W^{(2)}(S) \subset O(S)$ has finite index. We remark that this is parallel to Lemma 1.4, and is an important step to prove that log del Pezzo surfaces of index ≤ 2 are equivalent to DPN surfaces of elliptic type.

This finiteness was first observed and used for classification of hyperbolic lattices M with finite index $[O(M) : W^{(2)}(M)]$ in [Nik79], [Nik83]. We repeat arguments of [Nik79], [Nik83]. Let us take a general pair (X, θ) with $(S_X)_+ = S$. Then $S_X = S$, and the involution θ of X is unique by the condition that it is identical on $S_X = S$ and is -1 on the orthogonal complement to S_X in $H^2(X, \mathbb{Z})$. Thus, $\operatorname{Aut} X = \operatorname{Aut}(X, \theta)$. By Global Torelli Theorem for K3 (see [PS-Sh71]), the action of $\operatorname{Aut} X$ on S_X gives that $\operatorname{Aut} X$ and $O(S_X)/W^{(2)}(S_X)$ are isomorphic up to finite groups. In particular, they are finite simultaneously. Thus, $[O(S) : W^{(2)}(S)]$ is finite, if and only if $\operatorname{Aut}(X, \theta)$ is finite. If (X, θ) has elliptic type, then $\operatorname{Aut}(X, \theta)$ preserves X^θ and its component C_g with $(C_g)^2 > 0$. Since S_X is hyperbolic, it follows that the action of $\operatorname{Aut}(X, \theta)$ in S_X is finite. But it is known for K3 (see [PS-Sh71]) that the kernel of this action is also finite. It follows that $\operatorname{Aut}(X, \theta)$ and $[O(S) : W^{(2)}(S)]$ are finite. See more details on the results we used about K3 in Section 2.2.

Since $O(S)$ is arithmetic, $W^{(2)}(S)$ has a fundamental chamber $\mathcal{M}^{(2)}$ in $\mathcal{L}(S)$ of finite volume and with a finite number of faces (e.g. see [Vin85]). Since $W^{(2)}(S) \subset W^{(2,4)}(S) \subset O(S)$, the same is valid for $W^{(2,4)}(S)$.

Let $\mathcal{M}^{(2,4)} \subset \mathcal{L}(S)$ be a fundamental chamber of $W^{(2,4)}(S)$, and $\Gamma(P(\mathcal{M}^{(2,4)}))$ its Dynkin diagram (see [Vin85]). Vertices corresponding to different elements $f_1,\ f_2 \in P(\mathcal{M}^{(2,4)})$ are **not connected** by any edge, if $f_1 \cdot f_2 = 0$. They are connected by a **simple edge of the weight** m