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Log del Pezzo surfaces of index ≤ 2 and Smooth Divisor Theorem

1.1. Basic definitions and notation

Let Z be a normal algebraic surface, and K_Z be a canonical Weil divisor on it. The surface Z is called **$\mathbb{Q}$-Gorenstein** if a certain positive multiple of K_Z is Cartier, and **$\mathbb{Q}$-factorial** if this is true for any Weil divisor D. These properties are local: one has to require all singularities to be $\mathbb{Q}$-Gorenstein, respectively $\mathbb{Q}$-factorial.

Let us denote by $Z^1(Z)$ and $\mathrm{Div}(Z)$ the groups of Weil and Cartier divisors on Z. Assume that Z is $\mathbb{Q}$-factorial. Then the groups $Z^1(Z) \otimes \mathbb{Q}$ and $\mathrm{Div}(Z) \otimes \mathbb{Q}$ of $\mathbb{Q}$-Cartier divisors and $\mathbb{Q}$-Weil divisors coincide. The intersection form defines natural pairings

$$\mathrm{Div}(Z) \otimes \mathbb{Q} \times \mathrm{Div}(Z) \otimes \mathbb{Q} \to \mathbb{Q},$$

$$\mathrm{Div}(Z) \otimes \mathbb{R} \times \mathrm{Div}(Z) \otimes \mathbb{R} \to \mathbb{R}.$$

Quotient groups modulo kernels of these pairings are denoted $N_{\mathbb{Q}}(Z)$ and $N_{\mathbb{R}}(Z)$ respectively; if the surface Z is projective, they are finite-dimensional linear spaces. The **Kleiman–Mori cone** is a convex cone $\overline{\mathrm{NE}}(Z)$ in $N_{\mathbb{R}}(Z)$, the closure of the cone generated by the classes of effective curves.

Let D be a $\mathbb{Q}$-Cartier divisor on Z. We will say that D is ample if some positive multiple is an ample Cartier divisor in the usual sense. By *Kleiman's criterion* [Kle66], for this to hold it is necessary and sufficient that D defines a strictly positive linear function on $\overline{\mathrm{NE}}(Z) - \{0\}$.

One says that the **surface Z has only log terminal singularities** if it is $\mathbb{Q}$-Gorenstein and for one (and then any) resolution of singularities $\pi : Y \to Z$, in a natural formula $K_Y = \pi^* K_Z + \sum \alpha_i F_i$, where F_i are irreducible divisors and $\alpha_i \in \mathbb{Q}$, one has $\alpha_i > -1$. The least common multiple of denominators of α_i is called the **index** of Z.