CHAPTER VI

Invariance of plurigenera

§1. Background

A *deformation* (or a smooth deformation) of a compact complex manifold X is by definition a proper smooth surjective morphism $\pi \colon \mathcal{X} \to S$ of complex analytic varieties together with a point $s \in S$ such that the fiber $\mathcal{X}_s = \pi^{-1}(s)$ is isomorphic to X. The deformation is called projective if π is a projective morphism along X. A compact complex manifold is said to be *in the class* $\mathcal{C}$ if it is bimeromorphically equivalent to a compact Kähler manifold ([**18**], [**143**]). We are interested in the following:

1.1. Conjecture The m-genus $P_m(X) = \mathrm{h}^0(X, mK_X)$ is invariant under a deformation of a compact complex manifold in the class $\mathcal{C}$.

The deformation invariance of the plurigenera of compact complex surfaces was proved by Iitaka [**42**] by the classification theory of surfaces. Nakamura [**94**] gave a counterexample to the invariance in the case where X is not in the class $\mathcal{C}$. The invariance of the geometric genus $P_1(X) = p_g(X)$ for X in the class $\mathcal{C}$ is derived from the Hodge decomposition $\mathrm{H}^n(X, \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^q(X, \Omega_X^p)$ and the upper semi-continuity of $\mathrm{h}^q(X, \Omega_X^p)$. Levine [**75**] proved **1.1** for $m > 1$ in the case where mK_X is linearly equivalent to a reduced normal crossing divisor. Levine applied the Hodge theory to the cyclic covering branched along the divisor in order to show the existence of an infinitesimal lifting of a general section of $\mathrm{H}^0(X, mK_X)$.

A *degeneration* of compact complex manifolds is by definition a proper surjective morphism $\pi \colon \mathcal{X} \to S$ with connected fibers from a non-singular complex analytic variety into a non-singular curve that is smooth outside a given point $0 \in S$. We denote by $\mathcal{X}_t$ the scheme-theoretic fiber $\pi^{-1}(t)$. We say that a smooth fiber $\mathcal{X}_t$ ($t \neq 0$) degenerates into the special fiber $\mathcal{X}_0$. The degeneration is called projective if π is so. Let $\mathcal{X}_0 = \bigcup \Gamma_i$ be the irreducible decomposition of the special fiber. In the study of degeneration of algebraic surfaces (cf. [**15**]), the lower semi-continuity of the Kodaira dimension: $\kappa(\mathcal{X}_t) \geq \max \kappa(\Gamma_i)$ is expected to be true. However, there are counterexamples ([**108**], [**109**], [**140**], [**19**]) in the case where some Γ_i is not in the class $\mathcal{C}$. The following stronger conjecture is posed in [**98**]:

1.2. Conjecture If any irreducible component Γ_i of the special fiber $\mathcal{X}_0$ belongs to the class $\mathcal{C}$, then

$$P_m(\mathcal{X}_t) \geq \sum P_m(\Gamma_i)$$