CHAPTER V

Numerical Kodaira dimension

We give a criterion for an $\mathbb{R}$-divisor to be pseudo-effective in §**1** by applying the Kawamata–Viehweg vanishing theorem. In §**2**, we introduce two invariants, denoted by $\kappa_\sigma(D)$ and $\kappa_\nu(D)$, respectively, both of which seem to be the candidates and deserve to be called the numerical D-dimension for a pseudo-effective divisor D. Both invariants have many properties expected for numerical D-dimension, which we prove using the results in §**1**. In §**3**, we introduce the notion of ω-sheaves, which is useful for the study of direct images of relative pluricanonical sheaves. The notion of weak positivity introduced by Viehweg is refined also in §**3**. We prove some addition theorems for κ and κ_σ and for log-terminal pairs in §**4**. These are slight generalizations of Viehweg's results in [**147**]. In the last part of §**4**, we prove the abundance theorem in a special case where $\kappa_\sigma = 0$, as an application of the addition theorems.

§1. Pseudo-effective $\mathbb{R}$-divisors

§1.a. Base-point freeness.

1.1. Lemma *Let Δ and D be effective $\mathbb{R}$-divisors without common prime components on a normal variety X and let x be a point of X.*

1. *If (X, bD) and $(X, b/(b-1)\Delta)$ are log-terminal at x for some $b > 1$, then $(X, D + \Delta)$ is log-terminal at x.*
2. *Suppose that X is non-singular at x and $\operatorname{mult}_x \Delta < 1$. Then (X, Δ) is log-terminal at x.*
3. *Suppose that X is non-singular at x, (X, bD) is log-terminal at x, and $\operatorname{mult}_x \Delta < (b-1)/b$ for some $b > 1$. Then $(X, \Delta + D)$ is log-terminal at x.*

Proof. (1) Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety such that the union of the exceptional locus $G = \sum G_i$, the proper transform D_Y of D, and the proper transform Δ_Y of Δ is a simple normal crossing divisor. Then we can write

$$K_Y = f^*(K_X + bD) + \sum a_i G_i - bD_Y = f^*\left(K_X + \frac{b}{b-1}\Delta\right) + \sum c_i G_i - \frac{b}{b-1}\Delta_Y$$