CHAPTER II

Preliminaries

This chapter recalls some fundamental facts for the study of complex analytic and algebraic varieties. Some of them are well-known and we include no proofs. Some new notions and terminologies are introduced for the clarification of arguments in the subsequent chapters. We review some basic properties of complex analytic varieties in §**1**. The notion of divisor and some variants are explained in §**2**. The theory of linear systems is fundamental in the subject of algebraic geometry. Iitaka's theory of D-dimension has its base on the study of linear systems. We generalize the theories to those applicable to $\mathbb{R}$-divisors in §**3**, by using a result in Chapter **III**. Information most essential to a variety, such as Kodaira dimension, is usually derived from the information on the canonical divisor. The singularities appearing in the minimal model program for the birational classification of algebraic varieties are all related to some properties of the canonical divisor. They are the subjects of study in §**4**. Numerical properties of ample, nef, big, and pseudo-effective for $\mathbb{R}$-divisors are discussed in §**5**. Vanishing theorems related to the Kodaira vanishing are also mentioned. In §**6**, we recall such basics as Chern classes and semi-stability, indispensable for the study of vector bundles.

§1. Complex analytic varieties

§1.a. General theory. A *complex analytic space* X is a locally ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to the closed subspace of an open subset U of some complex affine space $\mathbb{C}^N$ defined as $X = \operatorname{Supp} \mathcal{O}_U/\mathcal{I} \subset U$ and $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I}|_X$ for a coherent $\mathcal{O}_U$-ideal sheaf $\mathcal{I}$. Here $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *coherent* if it satisfies the following conditions:

(1) It is finitely generated locally on X: For any point of X, there exist an open neighborhood U and a surjective homomorphism $\mathcal{O}_X^{\oplus k}|_U \to \mathcal{F}|_U$ for some $k \in \mathbb{N}$;
(2) For any homomorphism $\mathcal{O}_X^{\oplus l}|_U \to \mathcal{F}|_U$ over an open subset $U \subset X$, its kernel is finitely generated locally on U.

For a fixed complex analytic space X, a sheaf of $\mathcal{O}_X$-modules is called an $\mathcal{O}_X$-module, and a coherent $\mathcal{O}_X$-module is called simply a coherent sheaf. In this article, we always assume that complex analytic spaces are all Hausdorff and paracompact. We drop the words 'complex' and 'analytic' sometimes.