CHAPTER I

# Overview

## §1. Introduction

This article is based on the three unpublished preprints [**104**], [**105**], and [**106**] of the author. The main body is taken from [**104**], while Chapter **VI** is based upon [**105**] and Chapter **IV**, §**4** upon [**106**]. The way of unifying these preprints into one article, however, was not done in a simple manner. Some parts are moved around, although most things were kept mainly along the line of [**104**]. A new chapter II is added for the purpose of helping the reader understand this unification better. The main subjects of this article are:

- Zariski-decomposition problem;
- Numerical $D$-dimension;
- Addition theorem;
- Invariance of plurigenera.

**§1.a. Zariski-decomposition.** The theory of divisors plays an important role in algebraic geometry. Let $X$ be a normal complete algebraic variety defined over the complex number field $\mathbb{C}$ and let $D$ be a Cartier divisor. The complete linear system $|D|$, which is a projective space parametrizing all the effective divisors linearly equivalent to $D$, defines a rational map $\Phi_{|D|}\colon X \dashrightarrow |D|^\vee$ into the dual projective space $|D|^\vee$. The $D$-dimension $\kappa(D) = \kappa(D, X)$ is defined as the maximum of $\dim \Phi_{|mD|}(X)$ for $m > 0$ in the case: $|lD| \neq \emptyset$ for some $l > 0$. In the other case, i.e., $|lD| = \emptyset$ for any $l > 0$, we set $\kappa(D, X) = -\infty$ by definition. We have another expression for the $D$-dimension:

$$\kappa(D, X) = \begin{cases} -\infty, & \text{if } R(X, D) = \mathbb{C}\ , \\ \operatorname{tr.deg} R(X, D) - 1, & \text{otherwise,} \end{cases}$$

in terms of the graded ring

$$R(X, D) = \bigoplus\nolimits_{m \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(mD)).$$

The ring $R(X, D)$ is not always finitely generated as a $\mathbb{C}$-algebra. It is finitely generated if and only if there exist a birational morphism $\mu\colon Y \to X$ from a normal complete variety $Y$, a positive integer $m$, and an effective Cartier divisor $F$ of $Y$ such that

(1) $kF$ is the fixed divisor $|mk\mu^*D|_{\mathrm{fix}}$ for any $k > 0$,
(2) $\operatorname{Bs}|m\mu^*D - F| = \emptyset$.