# 7 Appendix

In this appendix, we present proofs of the propositions, which appeared in the previous sections. However, to prove them, we often need more fundamental results, for which we only give references. One of such results is the following "Area formula", which will be employed in sections 7.1 and 7.2. We refer to [9] for a proof of a more general Area formula.

| **Area formula** |
|---|
| $\left.\begin{array}{c}\xi\in C^1(\mathbf{R}^n,\mathbf{R}^n),\\ g\in L^1(\mathbf{R}^n),\\ A\subset\mathbf{R}^n \text{ measurable}\end{array}\right\} \Longrightarrow \int_{\xi(A)}\lvert g(y)\rvert dy\le\int_A \lvert g(\xi(x))\rvert\lvert\det(D\xi(x))\rvert dx$ |

We note that the Area formula is a change of variable formula when $|\det(D\xi)|$ may vanish. In fact, the equality holds if $|\det(D\xi)| > 0$ and $\xi$ is injective.

## 7.1 Proof of Ishii's lemma

First of all, we recall an important result by Aleksandrov. We refer to the Appendix of [6] and [10] for a "functional analytic" proof, and to [9] for a "measure theoretic" proof.

**Lemma 7.1.** (Theorem A.2 in [6]) *If $f : \mathbf{R}^n \to \mathbf{R}$ is convex, then for a.a. $x \in \mathbf{R}^n$, there is $(p, X) \in \mathbf{R}^n \times S^n$ such that*

$$f(x+h) = f(x) + \langle p, h\rangle + \frac{1}{2}\langle Xh, h\rangle + o(|h|^2) \quad \text{as } |h| \to 0.$$

*(i.e., $f$ is twice differentiable at a.a. $x \in \mathbf{R}^n$.)*

We next recall Jensen's lemma, which is a version of the ABP maximum principle in 7.2 below.

**Lemma 7.2.** (Lemma A.3 in [6]) *Let $f : \mathbf{R}^n \to \mathbf{R}$ be semi-convex (i.e. $x \to f(x) + C_0|x|^2$ is convex for some $C_0 \in \mathbf{R}$). Let $\hat{x} \in \mathbf{R}^n$ be a strict maximum point of $f$. Set $f_p(x) := f(x) - \langle p, x\rangle$ for $x \in \mathbf{R}^n$ and $p \in \mathbf{R}^n$.*

*Then, for $r > 0$, there are $C_1, \delta_0 > 0$ such that*

$$|\Gamma_{r,\delta}| \ge C_1\delta^n \quad \text{for } \delta \in (0, \delta_0],$$