6 L^p-viscosity solutions

In this section, we discuss the L^p-viscosity solution theory for uniformly elliptic PDEs:

$$F(x, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{6.1}$$

where $F : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given. Since we will use the fact that $u + C$ (for a constant $C \in \mathbf{R}$) satisfies the same (6.1), we suppose that F does not depend on u itself. Furthermore, to compare with classical results, we prefer to have the inhomogeneous term (the right hand side of (6.1)).

The aim in this section is to obtain the a priori estimates for L^p-viscosity solutions without assuming any continuity of the mapping $x \to F(x, q, X)$, and then to establish an existence result of L^p-viscosity solutions for Dirichlet problems.

Remark. In general, without the continuity assumption of $x \to F(x, p, X)$, even if $X \to F(x, p, X)$ is uniformly elliptic, we **cannot** expect the uniqueness of L^p-viscosity solutions. Because Nadirashvili (1997) gave a counterexample of the uniqueness.

6.1 A brief history

Let us simply consider the Poisson equation in a "smooth" domain Ω with zero-Dirichlet boundary condition:

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

In the literature of the regularity theory for uniformly elliptic PDEs of second-order, it is well-known that

$$\text{"if } f \in C^\sigma(\overline{\Omega}) \text{ for some } \sigma \in (0,1), \text{ then } u \in C^{2,\sigma}(\overline{\Omega})\text{".} \tag{6.3}$$

Here, $C^\sigma(U)$ (for a set $U \subset \mathbf{R}^n$) denotes the set of functions $f : U \to \mathbf{R}$ such that

$$\sup_{x \in U} |f(x)| + \sup_{x,y \in U, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\sigma} < \infty.$$