5 Generalized boundary value problems

In order to obtain the uniqueness of solutions of an ODE, we have to suppose certain initial or boundary condition. In the study of PDEs, we need to impose appropriate conditions on $\partial\Omega$ for the uniqueness of solutions.

Following the standard PDE theory, we shall treat a few typical boundary conditions in this section.

Since we are mainly interested in degenerate elliptic PDEs, we **cannot** expect "solutions" to satisfy the given boundary condition on the whole of $\partial\Omega$. The simplest example is as follows: For $\Omega := (0,1)$, consider the "degenerate" elliptic PDE

$$-\frac{du}{dx} + u = 0 \quad \text{in } (0,1).$$

Note that it is impossible to find a solution u of the above such that $u(0) = u(1) = 1$.

Our plan is to propose a definition of "generalized" solutions for boundary value problems. For this purpose, we extend the notion of viscosity solutions to possibly discontinuous PDEs on $\overline{\Omega}$ while we normally consider those in Ω.

For general $G : \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$, we are concerned with

$$G(x, u, Du, D^2u) = 0 \quad \text{in } \overline{\Omega}. \tag{5.1}$$

As in section 4.3, we define

$$G_*(x,r,p,X) := \liminf_{\varepsilon\to 0} \left\{ G(y,s,q,Y) \;\middle|\; \begin{array}{c} y \in \overline{\Omega} \cap B_\varepsilon(x), |s-r| < \varepsilon, \\ |q-p| < \varepsilon, \|Y - X\| < \varepsilon \end{array} \right\},$$

$$G^*(x,r,p,X) := \limsup_{\varepsilon\to 0} \left\{ G(y,s,q,Y) \;\middle|\; \begin{array}{c} y \in \overline{\Omega} \cap B_\varepsilon(x), |s-r| < \varepsilon, \\ |q-p| < \varepsilon, \|Y - X\| < \varepsilon \end{array} \right\}.$$

Definition. We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity subsolution (resp., supersolution) of (5.1) if, for any $\phi \in C^2(\overline{\Omega})$,

$$G_*(x, u^*(x), D\phi(x), D^2\phi(x)) \le 0$$

$$\big(\text{resp., } G^*(x, u_*(x), D\phi(x), D^2\phi(x)) \ge 0\big)$$

provided that $u^* - \phi$ (resp., $u_* - \phi$) attains its maximum (resp., minimum) at $\underline{x \in \overline{\Omega}}$.