4 Existence results

In this section, we present some existence results for viscosity solutions of second-order (degenerate) elliptic PDEs.

We first present a convenient existence result via Perron's method, which was established by Ishii in 1987.

Next, for Bellman and Isaacs equations, we give representation formulas for viscosity solutions. From the dynamic programming principle below, we will realize how natural the definition of viscosity solutions is.

4.1 Perron's method

In order to introduce Perron's method, we need the notion of viscosity solutions for semi-continuous functions.

Definition. For any function $u : \overline{\Omega} \to \mathbf{R}$, we denote the upper and lower semi-continuous envelope of u by u^* and u_*, respectively, which are defined by

$$u^*(x) = \lim_{\varepsilon \to 0} \sup_{y \in B_\varepsilon(x) \cap \overline{\Omega}} u(y) \quad \text{and} \quad u_*(x) = \lim_{\varepsilon \to 0} \inf_{y \in B_\varepsilon(x) \cap \overline{\Omega}} u(y).$$

We give some elementary properties for u^* and u_* without proofs.

Proposition 4.1. *For $u : \overline{\Omega} \to \mathbf{R}$, we have*

(1) $u_*(x) \le u(x) \le u^*(x)$ *for* $x \in \overline{\Omega}$,

(2) $u^*(x) = -(-u)_*(x)$ *for* $x \in \overline{\Omega}$,

(3) u^* *(resp., u_*) is upper (resp., lower) semi-continuous in* $\overline{\Omega}$, *i.e.* $\limsup_{y \to x} u^*(y) \le u^*(x)$, *(resp.,* $\liminf_{y \to x} u_*(y) \ge u_*(x)$*) for* $x \in \overline{\Omega}$,

(4) *if u is upper (resp., lower) semi-continuous in* $\overline{\Omega}$, *then* $u(x) = u^*(x)$ *(resp.,* $u(x) = u_*(x)$*) for* $x \in \overline{\Omega}$.

With these notations, we give our definition of viscosity solutions of

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega. \tag{4.1}$$

Definition. We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity subsolution (resp., supersolution) of (4.1) if u^* (resp., u_*) is a viscosity subsolution (resp., supersolution) of (4.1).