CHAPTER 11

Appendix

11.1. Existence of complements

PROPOSITION 11.1.1 ([**Sh3**]). *Let $f\colon X \to Z \ni o$ be a contraction from a surface and D a boundary on X such that $K_X + D$ is lc and $-(K_X + D)$ is f-nef and f-big. Then*

(i) *the linear system $|-m(K_X + D)|$ is base point free for some $m \in \mathbb{N}$;*
(ii) *$K_X + D$ is n-complementary near $f^{-1}(o)$ for some $n \in \mathbb{N}$;*
(iii) *the Mori cone $\overline{NE}(X/Z)$ is polyhedral and generated by irreducible curves.*

We hope that this fact has higher dimensional generalizations (cf. [**K3**], see also M. Reid's Appendix to [**Sh2**]).

PROOF. First we prove (i). We consider only the case of compact X. In the case $\dim Z \geq 1$ there are stronger results (see Theorem 6.0.6). Applying a log terminal modification 3.1.1, we may assume that $K_X + D$ is dlt (and X is smooth). Set $C := \lfloor D \rfloor$, $B := \{D\}$. Note that C is connected by Connectedness Lemma. Take sufficiently large and divisible $n \in \mathbb{N}$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-n(K_X + D) - C) \longrightarrow \mathcal{O}_X(-n(K_X + D)) \longrightarrow \mathcal{O}_C(-n(K_X + D)) \longrightarrow 0.$$

By Kawamata-Viehweg Vanishing [**KMM**, 1-2-6],

$$H^1(X, \mathcal{O}_X(-n(K_X + D) - C)) = H^1(X, \mathcal{O}_X(K_X + B - (n+1)(K_X + D))) = 0.$$

Therefore $C \cap \mathrm{Bs}|-n(K_X + D)| = \mathrm{Bs}\big|-n(K_X + D)|_C\big|$.

We claim that $\mathrm{Bs}\big|-n(K_X + D)|_C\big| = \varnothing$. Indeed, if C is not a tree of rational curves, then $p_a(C) = 1$ and C is either a smooth elliptic curve or a wheel of smooth rational curves (see Lemma 6.1.7). Moreover, $\mathrm{Supp}B \cap C = \varnothing$. But then $(K_X + D)|_C = (K_X + C)|_C = K_C = 0$ and $\mathrm{Bs}\big|-n(K_X + D)|_C\big| = \varnothing$ in this case. Note also that here we have an 1-complement by Lemma 8.3.8. Assume now that C is a tree of smooth rational curves. Then $\big|-n(K_X + D)|_{C_i}\big|$ is base point free on each component $C_i \subset C$ whenever $-n(K_X + D)$ is Cartier. Hence so is $\big|-n(K_X + D)|_C\big|$. This proves our claim.

Thus we have shown that $C \cap \mathrm{Bs}|-n(K_X+D)| = \varnothing$. Let $L \in |-n(K_X+D)|$ be a general member. Then $K_X + D + \frac{1}{n}L$ is dlt near C (see 1.3.2). By Connectedness