CHAPTER 10

On classification of exceptional complements: case $\delta \geq 1$

Now we study the case $\delta \geq 1$ in details.

10.1. The inequlity $\delta \leq 2$

In this section we show that $\delta \leq 2$. Replace (X, B) with a model $(\widetilde{X}, \widetilde{B})$. By construction, $\delta(X, B) = \delta(\widetilde{X}, \widetilde{B})$. Thus we assume that $\rho(X) = 1$, $B \in \Phi_{\mathbf{m}}$, $K_X + B$ is $(1/7)$-lt and $-(K_X + B)$ is nef. Moreover, there exists a boundary D defined by (9.1) such that $K_X + D$ is ample and lc. Let $C := \lfloor D \rfloor$. Then $\delta(X, B)$ is the number of components of C. Since $K_X + D$ is lc, C has only nodal singularities. The following is a very important ingredient in the classification.

THEOREM 10.1.1 ([**Sh3**]). *Notation as in 10. Then* $p_a(C) \leq 1$.

SKETCH OF PROOF. Assume that $p_a(C) \geq 2$. Consider the following birational modifications:

$$\begin{array}{ccccc} & & X^{\min} & & \\ & \swarrow^{\mu} & & \searrow^{\varphi} & \\ X & & & & X' \end{array} \qquad (10.1)$$

where $\mu\colon X^{\min} \to X$ be a minimal resolution and $\varphi\colon X^{\min} \to X'$ is a composition of contractions of -1-curves. Since $K_X + C$ is lc, C has only nodal singularities. By Lemma 9.1.8, X is smooth at Sing$C$. Therefore $C^{\min} \simeq C$. Thus $p_a(C) = p_a(C^{\min}) \geq 2$, $C^{\min}$ is not contracted and $p_a(C') \geq 2$. Take the crepant pull back

$$\mu^*(K_X + B) = K_{X^{\min}} + B^{\min}, \quad \text{with} \quad \mu_* B^{\min} = B$$

and put

$$B' := \varphi_* B^{\min}.$$

Note that both $-(K_{X^{\min}} + B^{\min})$ and $-(K_{X'} + B')$ are nef and big. Since $\rho(X) = 1$ and $C \simeq C^{\min}$, we have

(*) every two irreducible components of $C^{\min}$ intersect each other.

If $X' \simeq \mathbb{P}^2$, then $-(K_{X'} + \frac{6}{7}C')$ is ample. This gives $\frac{6}{7}\deg C' < 3$, $\deg C' \leq 3$ and $p_a(C') \leq 1$. Now we assume that $X' \simeq \mathbb{F}_n$. We claim that $n \geq 2$. Indeed, otherwise $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $X' \neq X^{\min}$ (because $\rho(X) = 1$) and we have at least