CHAPTER 9

Boundedness of exceptional complements

9.1. The main construction

In this chapter we discuss some boundedness results for exceptional log surfaces. The main results are Theorems 9.1.7, 9.1.11 and 9.1.12. Fix the following notation.

Let (X, B) be a projective log surface such that

(i) $K_X + B$ is lc;
(ii) $-(K_X + B)$ is nef;
(iii) the coefficients of B are standard or $d_i \geq 6/7$ (i.e. $B \in \Phi_{\mathbf{m}}$);
(iv) (X, B) is exceptional, i.e., any $\mathbb{Q}$-complement of $K_X + B$ is klt;
(v) there is a boundary $B^{\triangledown} \leq B$ such that $K_X + B^{\triangledown}$ klt and $-(K_X + B^{\triangledown})$ is nef and big.

By Corollary 8.4.2, (iv) is equivalent to

(iv)′ there are no regular nonklt complements.

Note also that by Theorem 8.3.1 and Corollary 8.3.2, $K_X + B$ is klt.

9.1.1. If $K_X + B$ is (1/7)-lt and $b_i < 6/7$, $\forall i$, then by Theorem 5.2.1, X belongs to a finite number of families. In this case $b_i \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}\}$, so (X, B) is bounded. Therefore we have a finite number of exceptional values of $\operatorname{compl}(X, B)$ in this case.

9.1.2. Now we assume that there is at least one (exceptional or not) divisor with $a(\cdot, B) \leq -1 + 1/7$. Following Shokurov [**Sh3**] we construct a model of (X, B) with $\rho = 1$. Similar birational modifications were used in [**KeM**] and was called the *hunt for a tiger*. Let $\mu\colon \widehat{X} \to X$ be the blowup of all exceptional divisors with $a(\cdot, B) \leq -6/7$ (see Lemma 3.1.9 and Proposition 3.1.2). Consider the crepant pull back

$$K_{\widehat{X}} + \widehat{B} = \mu^*(K_X + B), \quad \text{with} \quad \mu_* \widehat{B} = B.$$

Then $K_{\widehat{X}} + \widehat{B}$ is (1/7)-lt. By construction, $\widehat{B} \in \Phi_{\mathbf{m}}$. As in 8.3.3 we can construct a boundary $\widehat{B}^{\triangledown} \leq \widehat{B}$ on $\widehat{X}$ such that $K_{\widehat{X}} + \widehat{B}^{\triangledown}$ is klt and $-(K_{\widehat{X}} + \widehat{B}^{\triangledown})$ is nef and big. So on $\widehat{X}$ all our assumptions (i) – (v) hold and moreover,

(i)′ $K_{\widehat{X}} + \widehat{B}$ is (1/7)-lt and $\left\lfloor \frac{7}{6} \widehat{B} \right\rfloor \neq 0$.