CHAPTER 8

Inductive complements

8.1. Examples

Roughly speaking the main idea of this chapter is to discuss the following inductive statement:

> if a two-dimensional pair $(X/Z, D)$ is lc but not klt and $-(K_X + D)$ is nef over Z, then $K_X + D$ is 1, 2, 3, 4 or 6-complementary.

It is known that this assertion is true when $-(K_X + D)$ is big over Z (see Proposition 5.3.1) and in the local case. Unfortunately examples 8.1.1 and 8.1.2 below shows that in general, this is false and some additional assumptions are needed. The main result is the Inductive Theorem 8.3.1 which is a generalization of 5.3.1.

EXAMPLE 8.1.1 (**[Sh3]**). Let $\mathcal{E}$ be a indecomposable vector bundle of rank two and degree 0 over an elliptic curve Z. Then $\mathcal{E}$ is a nontrivial extension

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

(see e.g., **[Ha]**). Consider the ruled surface $X := \mathbb{P}_Z(\mathcal{E})$. Let $f\colon X \to Z$ be the projection and C a section corresponding to the above exact sequence. Then for the normal bundle of C in X we have $\mathcal{N}_{C/X} = \mathcal{O}_C$, hence $C|_C = 0$. In this situation we also have $-K_X \sim 2C$ (see **[Ha]**) and $(K_X + C)|_C = 0$. This yields $K_X|_C = 0$.

Since $\rho(X) = 2$, the Mori cone $\overline{NE}(X)$ is generated by two rays $R_1 = \mathbb{R}_+[F]$, where F is fiber of X and another ray, say R. Since $C^2 = 0$, C is nef and C generates R. In particular, both $-K_X$ and $-(K_X + C)$ are nef and numerically proportional to C.

We claim that $K_X + C$ is not n-complementary for any n. Indeed, otherwise we have $L \in |-m(K_X + C)|$ such that C is not a component of L. Then $L \cdot C = 0$ and $L \equiv mC$. The divisor $L - mC$ is trivial on fibers, hence $L - mC = f^*N$ for some $N \in \operatorname{Pic}(Z)$. Further, $C \cap L = \varnothing$. From this $(mC - L)|_C \sim 0$ (because $C|_C \sim 0$). Since $f|_C\colon C \to Z$ is an isomorphism, $f|_C^* N = (mC - L)|_C = 0$ gives $N \sim 0$, i.e., $L \sim mC$. Then the linear system $|L|$ determines on X a structure of an elliptic fibration $g\colon X \to \mathbb{P}^1$ with multiple fiber C. Hence $C|_C$ is an m-torsion element in $\operatorname{Pic}(C)$, a contradiction with $C|_C = 0$.

EXAMPLE 8.1.2. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We fix a projection $f\colon X \to \mathbb{P}^1$. Let C, H_1, H_2 be different sections of f and F_1, F_2, F_3 different fibers. Consider the log