CHAPTER 6

Birational contractions and two-dimensional log canonical singularities

THEOREM 6.0.6. *Let $(X/Z \ni o, D)$ be a log surface of local type, where $f\colon X \to Z \ni o$ is a contraction. Assume that $K_X + D$ is lc and $-(K_X + D)$ is f-nef and f-big. Then there exists an 1, 2, 3, 4, or 6-complement of $K_X + D$ which is not klt near $f^{-1}(o)$. Moreover, if there are no nonklt 1 or 2-complements, then $(X/Z \ni o, D)$ is exceptional. These complements $K_X + D^+$ can be taken so that $a(E, D) = -1$ implies $a(E, D^+) = -1$ for any divisor E of $\mathcal{K}(X)$.*

PROOF. Let H be an effective Cartier divisor on Z containing o and let $F := f^*H$. First we take the $c \in \mathbb{Q}$ such that $K_X + D + cF$ is maximally lc (see 5.3.3) and replace D with $D + cF$. This gives that $\mathrm{LCS}(X, D) \neq \varnothing$. Next we replace (X, D) with a log terminal modification. So we may assume that $K_X + D$ is dlt and $\lfloor D \rfloor \neq 0$. Then Proposition 4.4.3 and Theorem 4.1.10 give us that there exists a regular complement $K_X + D^+$ of $K_X + D$. By construction, $\lfloor D^+ \rfloor \geq \lfloor D \rfloor$. If $K_X + D$ is not exceptional, then there exists a $\mathbb{Q}$-complement $K_X + D'$ of $K_X + D$ and at least two divisors with discrepancy $a(\cdot, D') = -1$. Then we can replace D with D'. Taking a log terminal blowup, we obtain that $\lfloor D \rfloor$ is reducible. The rest follows by Theorem 4.1.10. □

COROLLARY 6.0.7. *Let (Z, Q) be a lc, but not klt two-dimensional singularity. Then the index of (Z, Q) is 1, 2, 3, 4, or 6.*

This fact has three-dimensional generalizations [**I**].

PROOF. Apply Theorem 6.0.6 to $f = \mathrm{id}$ and K_Z. We get an n-complement $K_Z + D$ with $n \in \{1, 2, 3, 4, 6\}$. Then $K_Z + D$ is lc and $n(K_Z + D) \sim 0$. But if $D \neq 0$, K_Z is klt (because $Q \in \mathrm{Supp} D$). □

COROLLARY 6.0.8. *Let $(X \ni P)$ be a normal surface germ. Let D be a boundary such that $D \in \Phi_{\mathbf{m}}$ and C a reduced Weil divisor on X. Assume that D and C have no common components. Then one of the following holds:*

(i) *$K_X + D + C$ is lc; or*
(ii) *$K_X + D + \alpha C$ is not lc for any $\alpha \geq 6/7$.*

Actually, we have more precise result 6.0.9. See [**Ko1**] for three-dimensional generalizations.