CHAPTER 5

Log del Pezzo surfaces

In the present chapter we discuss some properties of log del Pezzo surfaces.

5.1. Definitions and examples

DEFINITION 5.1.1. A projective log surface (X, D) is called

- a *log del Pezzo surface* if $K_X + D$ is lc and $-(K_X + D)$ is nef and big;
- a *log Enriques surface* if $K_X + D$ is lc and $K_X + D \equiv 0$.

Higher dimensional analogs of these are called *log Fano* and *log Calabi-Yau* varieties, respectively. Usually we omit D if $D = 0$.

If (X, D) is a log del Pezzo, then by Proposition 11.1.1 there exists some $\mathbb{Q}$-complement $K_X + D^+$ of $K_X + D$. The pair (X, D^+) is a log Enriques surface.

Examples of log del Pezzo surfaces are the classical ones, weighted projective planes $\mathbb{P}(a_1, a_2, a_3)$ with boundary $D = \sum d_i D_i$, where $D_i := \{x_i = 0\}$ and $\sum d_i < 3$, Hirzebruch surfaces $\mathbb{F}_n$ with boundary $\alpha\Sigma_0$, where Σ_0 is the negative section and $(n-2)/n \leq \alpha \leq 1$.

Let $f\colon (X', D') \to (X, D)$ be a birational log crepant morphism; that is,

$$K_{X'} + D' = f^*(K_X + D), \quad \text{with} \quad f_*D' = D.$$

Then (X, D) is a log del Pezzo if and only if so is (X', D') (see 1.1.5). Conversely, if $f\colon X' \to X$ is a birational morphism and (X', D') is a log del Pezzo then so is (X, f_*D'). Many examples can also be obtained by taking finite quotients; see 1.2.

EXAMPLE 5.1.2. Let $G \subset \mathrm{PGL}_2(\mathbb{C})$ be a finite subgroup, $X := \mathbb{P}^2/G$ and $f\colon \mathbb{P}^2 \to X$ the natural projection. As in 1.2, we define a boundary D on X by the condition $K_{\mathbb{P}^2} = f^*(K_X + D)$, where $D = \sum(1 - 1/r_i)D_i$, all the D_i are images of lines on $\mathbb{P}^2$, and r_i is the ramification index over D_i. For example, if G is the symmetric group $\mathfrak{S}_3$, acting on $\mathbb{P}^2$ by permutations of coordinates, X is the weighted projective plane $\mathbb{P}(1,2,3) = \mathrm{Proj}\mathbb{C}[\sigma_1, \sigma_2, \sigma_3]$, where the σ_i are the symmetric functions on coordinates on $\mathbb{P}^2$. The divisor D has exactly one component D_1 with coefficient $1/2$, where D_1 is determined by the equation

$$\sigma_1^2\sigma_2^2 - 4\sigma_2^3 - 4\sigma_1^3\sigma_3 - 27\sigma_3^2 + 18\sigma_1\sigma_2\sigma_3 = 0$$

(the equation of the discriminant). The surface X has exactly two singular points which are Du Val of types A_1 and A_2. Therefore X is a Gorenstein del Pezzo