CHAPTER 3

Log terminal modifications

3.1. Log terminal modifications

Many results of this chapter holds in arbitrary dimension modulo log MMP.

PROPOSITION-DEFINITION 3.1.1 (cf. [**Sh2**, 9.1]; see also [**Ut**, 6.16]). Let (X, D) be a log variety of dimension ≤ 3. Assume that $K_X + D$ is lc. Then there exists a *log terminal modification* of (X, D); that is, a birational contraction $g: X' \to X$ and a boundary D' on X' such that

(i) $K_{X'} + D' \equiv g^*(K_X + D)$;
(ii) $K_{X'} + D'$ is dlt;
(iii) X' is $\mathbb{Q}$-factorial.

Moreover, if $\dim X = 2$, it is possible to choose X' smooth.

PROOF. Consider a log resolution $h: Y \to X$. We have

$$K_Y + D_Y = h^*(K_X + D) + E^{(+)} - E^{(-)}, \tag{3.1}$$

where D_Y is the proper transform D on Y and $E^{(+)}$, $E^{(-)}$ are effective exceptional $\mathbb{Q}$-divisors without common components. Then $D_Y + E^{(-)}$ is a boundary and $K_Y + D_Y + E^{(-)}$ is dlt. Apply log MMP to $(Y, D_Y + E^{(-)})$ over X. We get a birational contraction $g: X' \to X$ from a normal $\mathbb{Q}$-factorial variety X'. Denote by D' the proper transform of $D_Y + E^{(-)}$ on X'. Then $K_{X'} + D'$ is dlt and g-nef. It is also obvious that $g_*D' = D$. We prove (i). Since the inverse to the birational map $h: Y \dashrightarrow X'$ does not contract divisors,

$$\begin{aligned} K_{X'} + D' = h_*\left(K_Y + D_Y + E^{(-)}\right) = h_*\left(f^*(K_X + D) + E^{(+)}\right) = \\ g^*(K_X + D) + h_*E^{(+)}. \end{aligned}$$

On the other hand, by numerical properties of contractions (see e.g., [**Sh2**, 1.1]) in the last formula all the coefficients of $h_*E^{(+)}$ should be nonpositive, i.e., all of them are equal to zero.

Finally, we consider the case $\dim X = 2$. If $E^{(+)} \neq 0$, then $(E^{(+)})^2 < 0$. From this $E^{(+)} \cdot E < 0$ for some E. Then $E^2 < 0$ and by (3.1), $K_Y \cdot E < 0$ and $(K_Y + D_Y + E^{(-)}) \cdot E < 0$. Hence E is a -1-curve and steps of log MMP over X are contractions of such curves. Continuing the process, we obtain a smooth surface X'. This proves the statement. □