CHAPTER 2

Inversion of adjunction

2.1. Two-dimensional toric singularities and log canonical singularities with a reduced boundary

2.1.1. If the cyclic group $\mathbb{Z}_m$ acts linearly on $\mathbb{C}^n$ by

$$x_1 \to \varepsilon^{a_1} x_1, \qquad x_2 \to \varepsilon^{a_2} x_2, \dots, x_n \to \varepsilon^{a_n} x_n,$$

where ε is a chosen primitive root of degree m of unity, we call the integers $a_1, \dots, a_n$ the *weights* of the action. In this case, the quotient is denoted by $\mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n)$. It is clear that the weights are defined modulo m and also depend on the choice of the primitive root ε.

Let (Z, Q) be a two-dimensional quotient singularity $\mathbb{C}^2/\mathbb{Z}_m(1, q)$, where $\gcd(q, m) = 1$ (in particular, this means that $\mathbb{Z}_m$ acts on $\mathbb{C}^2$ freely in codimension one). Then this singularity is toric, hence it is klt. The minimal resolution is obtained as a sequence of *weighted blowups* (see 3.2). The dual graph is a chain

$$\overset{-a_1}{\bigcirc} \text{---} \overset{-a_2}{\bigcirc} \text{---} \cdots \text{---} \overset{-a_{r-1}}{\bigcirc} \text{---} \overset{-a_r}{\bigcirc}, \qquad \text{with} \quad a_i \geq 2,$$

where the sequence $a_1, a_2, \dots, a_r$ is obtained from the continued fraction decomposition of m/q (see [**Hi**] or [**Br**]):

$$\frac{m}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots \cfrac{1}{a_r}}}. \tag{2.1}$$

Now we give the classification of two-dimensional log canonical singularities with nonempty reduced boundary, following Kawamata [**K**]. Note that this is much easier than the classification of all two-dimensional log canonical singularities.

THEOREM 2.1.2 ([**K**, 9.6], [**Ut**, ch. 3]). *Let $X \ni P$ be an analytic germ of a two-dimensional normal singularity and $X \supset C$ a (possibly reducible) reduced curve. Assume that $K_X + C$ is plt. Then*

$$(X, C) \simeq (\mathbb{C}^2, \{x = 0\})/\mathbb{Z}_m(1, a), \quad \textit{with} \quad \gcd(a, m) = 1.$$