CHAPTER 0

Introduction

These notes grew out of lectures I gave at Moscow University. Basically we follow works of V. V. Shokurov [**Sh2**], [**Sh3**]. These notes can help the reader to understand the main ideas of the theory. In particular, they can be considered as an introduction to [**Sh3**].

One of the basic problems in the modern birational minimal model program is the problem of constructing a divisor with rather "good" singularities in the anticanonical or multiple anticanonical linear system. Probably this question arises for the first time in the classification of Fano 3-folds (see e.g., [**Is**]) and was solved by V. V. Shokurov [**Sh**]. Later this technique was improved by Kawamata and others (see e.g., [**R**]). However, this method only applies to linear systems of Cartier divisors.

Another approach to constructing "good" divisors in the anticanonical linear system was proposed by Mori [**Mo**] in the proof of existence of three-dimensional flips. Unfortunately, now this method was applied only in analytic situation and in dimension 3 (see [**KoM**], [**P**] and also [**Ka**]).

The concept of complement was introduced by V. V. Shokurov in his proof of the existence of good divisors near nontrivial fibers of small contractions [**Sh2**]. Roughly speaking, an n-complement of the canonical divisor K_X is an element of the multiple anticanonical linear system $D \in |-nK_X|$ such that the pair $\left(X, \frac{1}{n}D\right)$ has only log canonical singularities (for precise definitions we refer to 4.1.3). Thus the question of the existence of complements can be posed for Fano or Calabi-Yau varieties (i.e., with numerically effective anticanonical divisor), and also for varieties with a fiber space structure on varieties of such types. For example, if on a smooth (or even with log canonical singularities) variety X the anticanonical divisor $-K_X$ is ample, then by Bertini's theorem there is an n-complement for some $n \in \mathbb{N}$. In case of Calabi-Yau varieties of the property of the canonical divisor K_X to be n-complementary is equivalent to triviality of nK_X. For example, it is known that for smooth surfaces such n can be taken in $\{1, 2, 3, 4, 6\}$.

These notes aim to do two things. The first is to give an introduction to the theory of complements, with rigorous proofs and motivating examples. For the first time the reader can be confused by rather tricky definition of n-complements (see (iii) of 4.1.3). However this definition is justified because of their useful properties – both birational and inductive (see 4.3 and 4.4). This shows that the property of