Chapter 10

Gauss-Manin Connections

10.1 Fibration

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential. We fix $\mathcal{A}$ in the rest of this section and write $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$. We saw in the previous section that B may be considered a moduli space of the family of essential simple affine ℓ-arrangements which are combinatorially equivalent to $\mathcal{A}$. Recall that $\mathbf{t}$ are homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. Let $\mathbf{u} = (u_1, \ldots, u_\ell)$ be standard coordinates for $\mathbb{C}^\ell$. Define

$$\mathsf{M} = \{(\mathbf{u}, \mathbf{t}) \in \mathbb{C}^\ell \times ((\mathbb{CP}^\ell)^*)^n \mid \mathbf{t} \in \mathsf{B},\ t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j \neq 0\ (i = 1, \ldots, n)\}.$$

Let

$$\pi : \mathsf{M} \longrightarrow \mathsf{B}$$

be the projection defined by $\pi(\mathbf{u}, \mathbf{t}) = \mathbf{t}$. Then the fiber $\mathsf{M}_{\mathbf{t}} = \pi^{-1}(\mathbf{t})$ is the complement of the affine arrangement $\mathcal{A}_{\mathbf{t}}$ whose hyperplanes are defined by $\alpha_i = t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j$ $(i = 1, \ldots, n)$. Thus $\pi : \mathsf{M} \longrightarrow \mathsf{B}$ is the complete family of essential simple affine arrangements in $\mathbb{C}^\ell$ which are combinatorially equivalent to $\mathcal{A}$. A result of Randell [Ra] implies that π is a fiber bundle over (the smooth part of) B.

Recall that d is the exterior differential operator with respect to the coordinates $\mathbf{u} = (u_1, \ldots, u_\ell)$ of $\mathbb{C}^\ell$ in the fiber, $\omega_i = d \log \alpha_i = d\alpha_i/\alpha_i$ for $1 \leq i \leq n$ and

$$\omega_\lambda = \sum_{i=1}^{n} \lambda_i \omega_i, \qquad \nabla_\lambda : \Omega_M^p \to \Omega_M^{p+1}, \qquad \nabla_\lambda \eta = d\eta + \omega_\lambda \wedge \eta.$$

In this section we compute covariant derivatives of differential forms in the fiber along the direction of the base.