Chapter 6

Bases

6.1 The Aomoto Complex

In this section we study the cohomology groups $H^p(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda \wedge)$. We do this by considering a universal complex whose specialization is the complex $(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda \wedge)$.

Definition 6.1.1 ([A1]). *Let $\mathbf{y} = \{y_H \mid H \in \mathcal{A}\}$ be a system of indeterminates in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. Let $\mathbb{C}[\mathbf{y}]$ be the polynomial ring in $\mathbf{y}$. Define a graded $\mathbb{C}[\mathbf{y}]$-algebra:*

$$\mathsf{A}^{\cdot}_{\mathbf{y}} = \mathsf{A}^{\cdot}_{\mathbf{y}}(\mathcal{A}) = \mathbb{C}[\mathbf{y}] \otimes_{\mathbb{C}} \mathsf{A}^{\cdot}(\mathcal{A}).$$

Let $a_{\mathbf{y}} = \sum_{H \in \mathcal{A}} y_H \otimes a_H \in \mathsf{A}^1_{\mathbf{y}}$. The complex $(\mathsf{A}^{\cdot}_{\mathbf{y}}(\mathcal{A}), a_{\mathbf{y}} \wedge)$

(1) $$0 \to \mathsf{A}^0_{\mathbf{y}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^1_{\mathbf{y}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \dots \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^r_{\mathbf{y}}(\mathcal{A}) \to 0$$

is called the **Aomoto complex**.

Let S be a multiplicative closed subset of $\mathbb{C}[\mathbf{y}]$. Consider the Aomoto complex of quotients by S

(2) $$0 \to \mathsf{A}^0_{\mathsf{S}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^1_{\mathsf{S}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \dots \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^r_{\mathsf{S}}(\mathcal{A}) \to 0,$$

where $\mathsf{A}^{\cdot}_{\mathsf{S}} = \mathsf{A}^{\cdot}_{\mathsf{S}}(\mathcal{A}) = \mathbb{C}[\mathbf{y}]_{\mathsf{S}} \otimes_{\mathbb{C}[\mathbf{y}]} \mathsf{A}^{\cdot}_{\mathbf{y}}(\mathcal{A})$.

Lemma 6.1.2. *If $\mathcal{C}$ is a nonempty central arrangement and Y is the multiplicative closed subset of $\mathbb{C}[\mathbf{y}]$ generated by $\sum_{H\in\mathcal{C}} y_H$; $\mathsf{Y} = \{(\sum_{H\in\mathcal{C}} y_H)^m \mid m \geq 0\}$, then the complex $(\mathsf{A}^{\cdot}_{\mathsf{Y}}(\mathcal{C}), a_{\mathbf{y}}\wedge)$ is acyclic.*

Proof. Let $\theta_E = \sum_{j=1}^{\ell} u_j(\partial/\partial u_j)$ be the Euler derivation. Denote the interior product by angle brackets. For $\eta \in \mathsf{A}^q_{\mathsf{Y}}(\mathcal{C})$, a standard formula [OT1, 4.73] gives $\langle\theta_E, a_{\mathbf{y}} \wedge \eta\rangle = \langle\theta_E, a_{\mathbf{y}}\rangle\eta - a_{\mathbf{y}}\langle\theta_E, \eta\rangle$, where $\langle\theta_E, a_{\mathbf{y}}\rangle = \sum_{H\in\mathcal{C}} y_H \otimes 1$. Thus if $a_{\mathbf{y}} \wedge \eta = 0$, the hypothesis gives $\eta = a_{\mathbf{y}} \wedge (\sum_{H\in\mathcal{C}} y_H)^{-1}\langle\theta_E, \eta\rangle$. □