Chapter 5

Combinatorics

5.1 The Orlik-Solomon Algebra

Let $E^1 = \oplus_{H\in\mathcal{A}}\mathbb{C}e_H$ and let $E = E(\mathcal{A}) = \Lambda(E^1)$ be the exterior algebra of E^1. If $|\mathcal{A}| = n$, then $E = \oplus_{p=0}^{n} E^p$, where $E^0 = \mathbb{C}$, E^1 agrees with its earlier definition and E^p is spanned over $\mathbb{C}$ by all $e_{H_1}\cdots e_{H_p}$ with $H_k \in \mathcal{A}$. Define a $\mathbb{C}$–linear map $\partial_E = \partial : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1$ and for $p \geq 2$

$$\partial(e_{H_1}\cdots e_{H_p}) = \sum_{k=1}^{p}(-1)^{k-1}e_{H_1}\cdots\widehat{e_{H_k}}\cdots e_{H_p}$$

for all $H_1,\ldots,H_p \in \mathcal{A}$. If $S = \{H_1,\ldots,H_p\}$, write $e_S = e_{H_1}\cdots e_{H_p}$, $\cap S = H_1 \cap\cdots\cap H_p$ and $|S| = p$. If $p = 0$, we agree that $S = \{\ \}$ is the empty tuple, $e_S = 1$, and $\cap S = V$. If $\cap S \neq \emptyset$, then we call S **dependent** when $r(\cap S) < |S|$ and **independent** when $r(\cap S) = |S|$. This agrees with linear dependence and independence of the hyperplanes in S.

Definition 5.1.1. *Let $\mathcal{A}$ be an affine arrangement. Let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ generated by*

$$\{e_S \mid \cap S = \emptyset\} \cup \{\partial e_S \mid S \text{ is dependent}\}.$$

The Orlik-Solomon algebra $\mathsf{A}(\mathcal{A})$ is defined by $\mathsf{A}(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$.

The grading of E induces a grading on A. The following basic properties of $\mathsf{A}^{\cdot}(\mathcal{A})$ will be needed in the sequel [OT1, 3.56, 3.72]:

Theorem 5.1.2. *(1) Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a deletion-restriction triple of a nonempty arrangement. Then there are exact sequences for $q \geq 0$:*

$$0 \to \mathsf{A}^q(\mathcal{A}') \to \mathsf{A}^q(\mathcal{A}) \to \mathsf{A}^{q-1}(\mathcal{A}'') \to 0.$$