Chapter 4

Non-Resonance Theorems

4.1 Logarithmic Complex

We review classical Hodge theory for an ℓ-dimensional compact complex projective manifold X. Let $\mathcal{O} = \mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on X and let $\Omega^{\cdot} = \Omega^{\cdot}_X$ be the de Rham complex of germs of holomorphic differential forms on X with the exterior differentials, where $\Omega^0 = \mathcal{O}$. Let $\mathbb{C} = \mathbb{C}_X$ denote the constant sheaf on X. It follows from the Poincaré Lemma that the sequence $0 \to \mathbb{C} \to \Omega^{\cdot}$ is exact. Let σ be the stupid filtration. The spectral sequence associated with the filtered complex $(\Omega^{\cdot}, \sigma)$, is:

$$E_1^{p,q} = H^q(X, \Omega^p) \Rightarrow E_\infty^{p+q} = H^{p+q}(X, \mathbb{C}).$$

Theorem 4.1.1 (Hodge). *This spectral sequence degenerates at the E_1 term. As a consequence, there is a decomposition*

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^q(X, \Omega^p)$$

which is called the **Hodge decompostion**.

Next recall "the non-compact version" or "the mixed version" of the Hodge decomposition. Let X be a complex quasiprojective manifold, a Zariski open set of a compact projective manifold. As a corollary of Hironaka's resolution theorem, we know that there exists a (smooth) projective manifold $\bar{X}$ such that $Y = \bar{X} \setminus X$ is a normal crossing divisor. Each $x \in \bar{X}$ has a coordinate neighborhood V_x with coordinate system $(z_1, z_2, \dots, z_\ell)$ and an integer k $(0 \leq k \leq \ell)$ such that $z_1(x) = z_2(x) = \cdots = z_\ell(x) = 0$ and Y is defined locally by the equation $z_1 z_2 \dots z_k = 0$.

Definition 4.1.2. *For each $x \in \bar{X}$ and $p \geq 0$, define the $\mathcal{O}_x$–module*

$$\Omega^p(\log Y)_x = \{\omega \mid \ \omega \textit{ is the germ of a meromorphic } p\textit{–form such that } (z_1 z_2 \dots z_k)\omega \in \Omega^p \textit{ and } (z_1 z_2 \dots z_k) d\omega \in \Omega^{p+1}\}.$$