12 Applications to semilinear wave and Klein - Gordon equations

12.1 Application to semilinear wave equation

In this Chapter we consider semilinear wave equation

$$\square u = F_\lambda(u) \tag{12.1.1}$$

in $\mathbf{R}^{n+1}$. Here the nonlinearity $F_\lambda(u)$ is a C^1 function of u for any real $\lambda > 1$ so that the following estimate

$$\left|\left(\frac{\partial}{\partial u}\right)^j F_\lambda(u)\right| \leq C|u|^{\lambda-j} \quad , \quad j = 0, 1 \tag{12.1.2}$$

is fulfilled for u close to zero. Here the constant C may depend on j, λ, but C is independent of u. A typical model is $F_\lambda(u) = |u|^\lambda$. If $f, g \in C_0^\infty(\mathbf{R}^n)$ are fixed, we shall consider the corresponding Cauchy problem for (12.1.1) with initial data

$$u(0,x) = \varepsilon f(x), \ \partial_t u(0,x) = \varepsilon g(x). \tag{12.1.3}$$

For $\varepsilon > 0$ small enough solutions of the Cauchy problem (12.1.1) and (12.1.3) are called small amplitude solutions. Instead of fixing the functions f, g and taking $\varepsilon > 0$ small enough we could take initial data of the form

$$u(0,x) = f(x), \ \partial_t u(0,x) = g(x) \tag{12.1.4}$$

and assume that suitable Sobolev norms of the initial data are sufficiently small. All results of this section could be formulated for these small data solutions, but for sake of simplicity we shall consider only the case of small amplitude solutions.

Our main goal then is to find, for a given n, the sharp range of powers for which one always has a global weak solution of (12.1.1), (12.1.3), if $\varepsilon > 0$ is small enough.

Note that, even in the linear case, where one solves an inhomogeneous equation with a Lipschitz forcing term, in general one can only obtain weak solutions.

Let us now give some historical background. In 1979, John [26] showed that when $n = 3$ global solutions always exist if $\lambda > 1 + \sqrt{2}$ and $\varepsilon > 0$ is small. He also showed that the power $1 + \sqrt{2}$ is critical in the sense that no such result can hold if $\lambda < 1 + \sqrt{2}$ and $F_\lambda(u) = |u|^\lambda$. It was shown sometime later by Schaeffer [47] that there can also be blowup for arbitrarily small data in $(1+3)$-dimensions when $\lambda = 1 + \sqrt{2}$.

The number $1+\sqrt{2}$ appears to have first arisen in Strauss' work [53] on scattering for small-amplitude semilinear Schrödinger equations. Based on this, he made the conjecture in [54] that when $n \geq 2$ global solutions of (12.1.1),(12.1.3) should always exist if ε is small and λ is greater than a critical power which is the solution of the quadratic equation

$$(n-1)\lambda_c^2 - (n+1)\lambda_c - 2 = 0, \ \lambda_c > 1. \tag{12.1.5}$$