11 A priori estimates for the wave equation

11.1 Statement of the main weighted estimates for inhomogeneous wave equation

The representation of solution of the inhomogeneous problem (10.2.1) have been expressed by the formula (10.2.21) involving the operator $T_{\rho,\sigma}$. This is an operator acting on functions f on X by the formula

$$T_{\rho,\sigma}(f)(\Omega) = \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \int_0^\infty \frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda} P_\lambda(f)(\Omega) d\lambda. \tag{11.1.1}$$

Our main estimate for this operator is given in the following.

Theorem 11.1.1 *Let $\rho \geq 4\sigma \geq 1$ and $f \in S(X)$. Then for*

$$\frac{n-1}{2(n+1)} \leq \frac{1}{q} \leq \frac{1}{2}\ , \ \frac{1}{p} = 1 - \frac{1}{q}$$

we have the estimate

$$\|T_{\rho,\sigma}(f)\|_{L^q(X)} \leq C \ln(\rho/\sigma) \frac{\sigma^A}{\rho^B} \|f\|_{L^p(X)}, \tag{11.1.2}$$

where

$$A = 1 + B\ , \ B = \frac{n-1}{p}.$$

Assuming the supports of u and F are in the light cone, we can use the coordinates

$$\rho = (t^2 - |x|^2)^{1/2}, \Omega = (t,x)/\rho \in X,$$

and we can represent the L^q-norm in the form have

$$\|\rho^\alpha u\|_{L^q(\mathbf{R}^{n+1})} = \left(\int_0^\infty \|\rho^\alpha u(\rho .)\|^q_{L^q(X)} \rho^n d\rho \right)^{1/q}. \tag{11.1.3}$$

Next step is to use the trivial inequality

$$\|F\|_{L^q(\mathbf{R}^{n+1})} \leq C \sup_{\rho>0} \rho^{(n+1)/q+\varepsilon} \|F(\rho\ .)\|_{L^q(X)} \tag{11.1.4}$$

Thus the right side of (11.1.3) can be estimated from above by constant times

$$\sup_{\rho>0} \rho^{\alpha+(n+1)/q+\varepsilon} \|u(\rho\ .)\|_{L^q(X)}$$