10 Fundamental solutions

10.1 Cauchy problem for homogeneous wave equation

Our purpose in this section is to construct a solution of the problem

$$\Box E = 0,$$

$$E(0,x) = 0\ ,\ \partial_t E(0,x) = \delta(x), \tag{10.1.1}$$

where

$$\Box = -\partial_t^2 + \Delta.$$

Once the solution $E = E(t,x)$ is found, one can represent the solution of

$$\Box u = 0,$$

$$u(0,x) = 0\ ,\ \partial_t u(0,x) = f(x) \tag{10.1.2}$$

by

$$u = E(t,.) * f. \tag{10.1.3}$$

Since

$$\hat{u}(t,\xi) = \hat{E}(t,\xi)\hat{f}(\xi),$$

comparing the representation of u with (3.3.7), we see that

$$\hat{E}(t,\xi) = \frac{\sin(t|\xi|)}{|\xi|}. \tag{10.1.4}$$

To construct fundamental solution we define s_+^{-z} for any complex number z with Re$z < 1$ by

$$s_+^{-z} = \begin{cases} s^{-z} & \text{if } s > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.5}$$

It is clear that this is a classical function in $L^1_{loc}(\mathbf{R})$ for Re$z < 1$. Note that

$$\frac{d}{ds} s_+^{-z} = -z\ s_+^{-z-1}, \quad \text{for Re } z < 0. \tag{10.1.6}$$

The above relation enables one to extend the definition of s_+^{-z} for $1 \le$ Re$z < 2$. Namely, we define (for $1 \le$ Re$z < 2$)

$$s_+^{-z} = \frac{1}{(-z+1)} \frac{d}{ds}\left(s_+^{-z+1}\right), \tag{10.1.7}$$

where the derivative in the right side is taken in the sense of distributions. Moreover for $k \le$ Re$z < k+1$ we define s_+^{-z} by the relation

$$\begin{aligned} s_+^{-z} &= \frac{1}{(-z+1)...(-z+k)} \left(\frac{d}{ds}\right)^k \left(s_+^{-z+k}\right) \\ &= \frac{\Gamma(-z+1)}{\Gamma(-z+k+1)} \left(\frac{d}{ds}\right)^k s_+^{-z+k}, \end{aligned} \tag{10.1.8}$$