9 Sobolev spaces on manifolds with constant negative curvature

9.1 Sobolev spaces on the upper branch of the hyperboloid

The upper branch of the unit hyperboloid

$$X: \ t = \sqrt{|x|^2 + 1}$$

is a Riemannian manifold with constant negative curvature $K(x) = -1$. With respect to the parametrization of X

$$\Omega = (\mathrm{ch} r, \omega \ \mathrm{sh} r) \in X, \tag{9.1.1}$$

where $r > 0, \omega \in \mathbf{S}^{n-1}$ the metric on X is

$$ds^2 = dr^2 + \mathrm{sh}^2 r d\omega^2,$$

where $d\omega^2$ is the standard metric on $\mathbf{S}^{n-1}$. In particular, if $f : X \to \mathbb{R}$ and $d\Omega$ is the standard measure on X, we have

$$\int_X f(\Omega) d\Omega = \int_0^\infty \int_{\mathbf{S}^{n-1}} f(\mathrm{ch} r, \omega \mathrm{sh} r)(\mathrm{sh} r)^{n-1} dr d\omega.$$

With respect to this parametrization, the Laplace-Beltrami operator on X takes the form (8.1.18).

If f is a real integrable function defined on X, then we observe that

$$\int_X f(\Omega) d\Omega = \int_{\mathbb{R}^n} f(\langle x \rangle, x) \frac{dx}{\langle x \rangle}, \tag{9.1.2}$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$.

Further, in the interior of the positive light cone

$$\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \ : \ |x| < t\}, \tag{9.1.3}$$

one can introduce the coordinates

$$\rho = \sqrt{t^2 - |x|^2}, \quad \Omega = \left(\frac{t}{\rho}, \frac{x}{\rho}\right) \in X$$

obtaining the following decomposition of the D'Alembertian operator(see Lemma 8.2.1)

$$\square = -\partial_t^2 + \Delta$$

$$\rho^2 \square = -(\rho\partial_\rho)^2 - (n-1)\rho\partial_\rho + \Delta_X,$$

$$\square = -\partial_\rho^2 - \frac{n}{\rho}\partial_\rho + \frac{a^2}{\rho^2}\Delta_{X_a},$$