8 Fourier transformation on manifolds with constant negative curvature

8.1 Models of manifolds with constant negative curvature

Our goal is to extend the study of the Fourier transform and Sobolev spaces to the case of a manifold of constant negative curvature. For the purpose we introduce initially some typical models of manifolds with curvature -1. For more details on the subject one can see the book of S.Helgason [20], where general symmetric spaces are studied. Probably, manifolds with curvature -1 are simplest case of symmetric spaces of rank 1. That is why we shall concentrate our attention to this case. Our secondary purpose shall be to see the case of constant curvature $-a$. Introducing the Fourier and inverse Fourier transform for this case, we would want to see the case of flat Euclidean space as a limiting case $a \to 0$.

We turn to the models of manifold with curvature $-a, a > 0$.

Example 1. For any surface $S \subset \mathbf{R}^{n+1}$ defined by

$$S: \ t = \psi(x), x = (x_1, ..., x_n) \in \mathbf{R}^n \tag{8.1.1}$$

the Minkowski metric

$$dl^2 = -dt^2 + dx^2 \tag{8.1.2}$$

induces on S a Riemannian metric provided S is spacelike, i.e. any vector tangential to S is spacelike with respect to the form (8.1.2). More precisely, the metric induced by the embedding $S \subset \mathbf{R}^{n+1}$ is

$$ds^2 = (dl_{|S})^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j, \tag{8.1.3}$$

where

$$g_{ij} = \delta_{ij} - \frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}. \tag{8.1.4}$$

One can check that

$$g = \det(g_{ij}) = 1 - |\nabla\psi|^2$$

and the condition that S is spacelike means that $g(x) > 0$. The unit vector normal to S at the point $(x, \psi(x)) \in S$ is

$$N(x) = \frac{(1, \nabla\psi(x))}{\sqrt{1 - |\nabla\psi|^2}}.$$

The second quadratic form on S is

$$\sum_{i,j=1}^{n} b_{ij} dx^i dx^j,$$