7 Weighted Sobolev spaces on flat space

7.1 Abstract localized norms

Definition 7.1.1 *Let A be a Banach space with norm $\|\cdot\|_A$. A* Paley-Littlewood partition of identity *(PL partition for short) is a sequence $\pi = \{\pi_j\}_{j\geq 0}$ of bounded operators on A such that: the series $\sum_{j\geq 0}\pi_j$ converges strongly (i.e., pointwise) to the identity operator on A, and in addition there exists an integer $N \geq 1$ such that*

$$\pi_j\pi_k = 0 \ \ for \ \ |j-k| \geq N. \tag{7.1.1}$$

Remark 7.1.1 *We shall frequently encounter the following situation: we have two real valued functions $\phi(x)$ and $\psi(x)$ defined on some domain D, so that there exists a constant $C > 0$ such that for all $x \in D$*

$$C^{-1}\phi(x) \leq \psi(x) \leq C\phi(x). \tag{7.1.2}$$

In such cases we shall say that ϕ and *ψ* are equivalent on *D, and we shall write*

$$\phi(x) \sim \psi(x) \tag{7.1.3}$$

for $x \in D$.

Example 7.1.1 *Let $\{\phi_j\}_{j\geq 0}$ be a Paley-Littlewood partition of unity on $\mathbb{R}^n$, i.e., a sequence $\phi_j \in C_c^\infty(\mathbb{R}^n)$ such that $\phi_j \geq 0$, $\sum \phi_j = 1$, and*

$$\operatorname{supp}\phi_0 \subseteq \{|x| \leq 2\}, \qquad \operatorname{supp}\phi_j \subseteq \{2^{j-1} \leq |x| \leq 2^{j+1}\} \ \ j \geq 1. \tag{7.1.4}$$

More precisely, fix an arbitrary nonnegative $\psi \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \psi \leq 1$, equal to 1 on the ball $B(0,1/2)$ and vanishing outside $B(0,1)$, and define

$$\phi(x) = \psi(x/2) - \psi(x), \quad \phi_0(x) = \psi(x/2), \quad \phi_j(x) = \phi(2^{-j}x), \ j \geq 1. \tag{7.1.5}$$

This gives a partition of unity satisfying 7.1.4, and we shall call it a (standard) Paley-Littlewood partition of unity (PL partition for short).

We remark that if we choose $A = L^p(\mathbb{R}^n)$, $p \in [1,\infty]$, and define $\pi_j : A \to A$ as the multiplication operator by ϕ_j then $\pi = \{\pi_j\}$ is a PL partition of identity in the sense of Definition 7.1.1. Moreover, it satisfies the following important property, which will be used several times in the sequel: for any $1 \leq p < \infty$

$$\|u\|^p_{L^p(\mathbb{R}^n)} \sim \sum_{j\geq 0} \|\phi_j u\|^p_{L^p(\mathbb{R}^n)} \tag{7.1.6}$$

and similarly

$$\|u\|_{L^\infty(\mathbb{R}^n)} \sim \sup_{j\geq 0} \|\phi_j u\|_{L^\infty(\mathbb{R}^n)}.$$