6 Complex interpolation and fractional Sobolev spaces on flat space

6.1 Abstract complex interpolation for couple of Banach spaces

(see [2], [62])

For any couple $\bar{A} = (A_0, A_1)$ of Banach spaces A_0, A_1 we denote by $\Sigma(\bar{A})$ and by $\Delta(\bar{A})$ their sum and intersection respectively, i.e.

$$\Sigma(\bar{A}) = A_0 + A_1 \ , \ \Delta(\bar{A}) = A_0 \bigcap A_1 \tag{6.1.1}$$

with norms

$$\begin{gathered} \|a\|_{\Sigma(\bar{A})} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} \ ; a = a_0 + a_1 \ , a_0 \in A_0, a_1 \in A_1\} \\ \|a\|_{\Delta(\bar{A})} = \max(\|a\|_{A_0}, \|a\|_{A_1}). \end{gathered} \tag{6.1.2}$$

Then $\Sigma(A_0, A_1)$ and $\Delta(A_0, A_1)$ are Banach spaces.

The complex interpolation for the couple $\bar{A} = (A_0, A_1)$ can be associated with the space $F(\bar{A})$ of functions $f(z)$ defined, bounded and continuous in the strip

$$S = \{z \in \mathbf{C}; 0 \leq \mathrm{Re} z \leq 1\}$$

with values in $\Sigma(\bar{A})$ and satisfying the properties

$$f(it) \in A_0, \ t \in \mathbf{R} \tag{6.1.3}$$

$$f(1 + it) \in A_1, \ t \in \mathbf{R} \tag{6.1.4}$$

$$f : S_0 = \{z \in \mathbf{C}; 0 < \mathrm{Re} z < 1\} \ \rightarrow \ \Sigma(\bar{A}) \ \text{ is holomorphic.} \tag{6.1.5}$$

Then $F(\bar{A})$ is a Banach space with norm

$$\|f\|_F = \max\left(\sup_{t\in\mathbf{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbf{R}} \|f(1 + it)\|_{A_1}\right).$$

To show this we apply three lines lemma (see Lemma 3.2.1) with $\gamma = 0$ and see that $\|f\|_F = 0$ implies $f(z) = 0$ for $z \in S$.

To show that $F(\bar{A})$ is a Banach space, we take a Cauchy sequence

$$\{f_k(z)\}_{k=1}^{\infty}, f_k \in F(\bar{A}).$$

Then for $j = 0, 1$ and for $t \in \mathbf{R}$ fixed the sequence $f_k(j + it)$ tends to an element in A_j and we denote this element by $f(j + it)$. In a standard way, we see that $f_k(j + it)$ converges uniformly on $\mathbf{R}$ to $f(j + it)$.