5 Stationary phase method and pseudodifferential operators

5.1 Stationary phase method

In this section we shall give a brief review of the methods to study the asymptotic behavior of oscillatory integrals of type

$$I(R) = \int_{\mathbf{R}^n} e^{iR\phi(x)} f(x) dx \tag{5.1.1}$$

as $R > 0$ tends to infinity.

Here $f(x), \phi(x)$ are smooth functions defined on $\mathbb{R}^n$ with $\phi(x)$ being real-valued.

First, we consider the case, when the phase function $\varphi(x)$ has no critical points. More precisely, we consider the case, when there exist $\delta > 0, \delta \leq 1$ and $C > 0$ so that

$$|\nabla\phi(x)| \geq C^{-1} < x >^{\delta}, \quad < x >^2 = 1 + |x|^2, \tag{5.1.2}$$

$$|\partial_x^\alpha \nabla\phi(x)| \leq C < x >^{\delta - |\alpha|} \tag{5.1.3}$$

for any $x \in \operatorname{supp} f$.

Lemma 5.1.1 *Suppose the assumptions (5.1.2), (5.1.3) are fulfilled and $f(x)$ is a smooth function with compact support. Then for any integer $N \geq 0$ and for any $\varepsilon > 0$ we have*

$$|I(R)| \leq \frac{C}{R^N} \sum_{|\alpha| \leq N} \| < x >^{-N\delta - N + |\alpha| + n/2 + \varepsilon} \partial^\alpha f \|_{L^2(\mathbb{R}^n)}.$$

Proof. Given any first order differential operator

$$L(x, \partial_x) = (\sum_{j=1}^n a_j(x) \partial_{x_j}) + b(x),$$

we denote by L^* its adjoint operator with respect to the inner product in $L^2(\mathbb{R}^n)$, i.e.

$$L^*(x, \partial_x) = -(\sum_{j=1}^n \overline{a_j(x)} \partial_{x_j}) + \overline{b(x)} + \sum_{j=1}^n \partial_{x_j} \overline{a_j(x)}.$$

Therefore, for any couple f, g of smooth compactly supported functions on $\mathbb{R}^n$ we have

$$(Lf, g)_{L^2(\mathbb{R}^n)} = (f, L^* g)_{L^2(\mathbb{R}^n)}. \tag{5.1.4}$$

Let $L(x, \partial_x)$ be the differential operator, such that its adjoint is

$$L^* = i^{-1} \sum_{k=1}^n \frac{\partial_{x_k}\phi}{|\nabla\phi|^2} \partial_{x_k},$$