4 Main hyperbolic equations and energy type estimates

4.1 Linear wave and Klein-Gordon equations

Our first step in this chapter is to formulate some of the most important hyperbolic equations in mathematical physics.

In these lectures we shall focus our attention mainly to the wave and Klein-Gordon equations as the basic examples of hyperbolic equations in mathematical physics.

The wave equation is an important problem in continuum mechanics. A derivation of this equation in the model of vibrating string can be found in [59] Chapter 2.

The same equation plays a crucial role in relativistic quantum mechanics, since it is connected with a model of a massless relativistic field $u = u(t, x)$, where t is the time variable and

$$x = (x_1, ..., x_n) \in \mathbf{R}^n$$

are the space variables. The wave equation satisfied by the field u has the form

$$(-\partial_t^2 + \Delta)u = F, \tag{4.1.1}$$

where

$$\Delta = \partial_{x_1}^2 + ... + \partial_{x_n}^2$$

is the Laplace operator and $F = F(t, x)$ is a given known function. Usually, the operator

$$\square = -\partial_t^2 + \Delta$$

is called D'Alembert operator.

For the case, when a scalar relativistic field has a mass, the corresponding equation is called Klein-Gordon equation and this equation has the form

$$(-\partial_t^2 + \Delta - M^2)u = F, \tag{4.1.2}$$

where $M > 0$ is the mass of the field.

In general we can consider the wave equation as a partial case of Klein-Gordon equation with mass zero.

The first important physical law for these equation is the conservation of energy, when the external force F is identically zero.

Indeed, let us assume the solution is smooth and for any fixed t has a compact support. Then multiplying (4.1.2) by $\partial_t u$ we see that the energy

$$E(t) = \frac{1}{2} \int |\partial_t u(t, x)|^2 + \\ + |\nabla_x u(t, x)|^2 + M|u(t, x)|^2 dx \tag{4.1.3}$$