1 Introduction

The main purpose of these lecture notes is to represent some new ideas for the study of semi linear hyperbolic equations .

A fruitful classical tool to establish existence and to study the qualitative properties of the solutions of hyperbolic equations are the Sobolev spaces $W_p^s(\mathbf{R}^n)$. For integer values of $s \in \mathbf{R}$ the norm in this space is defined by

$$\|f\|_{W_p^s(\mathbf{R}^n)} = \sum_{|\alpha|\leq s} \|\partial_x^\alpha f\|_{L^p(\mathbf{R}^n)}, \tag{1.0.1}$$

while for $s \geq 0$ fractional the definition of W_p^s needs interpolation methods.

To obtain existence of a local solution to corresponding nonlinear problems one can combine classical Sobolev embedding with the energy conservation law. For the important problem, when one looks for local solutions with minimal regularity properties, it is necessary to use Sobolev spaces of fractional order ([3], [16], [17], [40]).

In these lectures we shall focus our attention to two basic hyperbolic equations - wave equation and Klein – Gordon equation. They have an additional symmetry, namely they are invariant under space - time rotations. This fact implies that we can use the usual derivatives

$$\partial_t, \partial_{x_1}, ..., \partial_{x_n} \tag{1.0.2}$$

in combination with the following vector fields

$$\begin{aligned} Y_{jk} &= x_j\partial_{x_k} - x_k\partial_{x_j},\ j,k = 1,...,n, \\ Y_{0j} &= t\partial_{x_j} + x_j\partial_t,\ j = 1,...,n. \end{aligned} \tag{1.0.3}$$

Modifying the definition of norm in Sobolev space (1.0.1) as follows

$$\|f\|_{W_p^s(\mathbf{R}^n,Y)} = \sum_{|\alpha|+|\beta|\leq s} \|\partial_x^\alpha Y^\beta f\|_{L^p(\mathbf{R}^n)}, \tag{1.0.4}$$

for the case $s \geq 0$ an integer, one can treat the problem of existence of global solutions with small initial data. This approach was developed successfully by S.Klainerman [29] , L.Hörmander [24] and was applied to other important equations and systems of mathematical physics.

As far as we know, there is no satisfactory theory of these modified spaces for fractional values of s.

Having in mind the importance of classical Sobolev spaces of fractional order, we shall represent here an approach to introduce and study Sobolev spaces associated with vector fields (1.0.3) for the case of fractional order s.

This first new idea is based on the application of Sobolev and weighted Sobolev spaces on the manifold

$$M = \{t^2 - |x|^2 = \pm 1\}.$$