Chapter 6

Limits of Metrics Spaces

A key ingredient in the proof of the Orbifold Theorem is the analysis of limits of metric spaces. In this chapter we give a short account of Gromov's theory of limits of metric spaces as re-interpreted using ϵ-approximations by Thurston. See Gromov's book [32] for a detailed treatment with many interesting applications.

Roughly, there is an "ϵ-approximation" between two metric spaces if the spaces look the same if we ignore details of size ϵ or smaller. From this we define the *Gromov-Hausdorff distance* between two compact metric spaces, and convergence of sequences of metric spaces.

This generalizes the classical notion of *Hausdorff distance* between two subsets A, B of a metric space X:

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subset N(B, \epsilon; X) \text{ and } B \subset N(A, \epsilon; X)\},$$

where

$$N(A, r; X) = \{\ x \in X\ :\ \exists a \in A\ \ d(x, a) < r\ \}$$

denotes the (open) neighbourhood of radius r around A in X. (See [5] for a detailed discussion of the Hausdorff distance and many geometric applications).

The *Gromov-Hausdorff distance* also generalizes the notion of *Lipschitz distance* between homeomorphic metric spaces. A bijection $f : X \longrightarrow Y$ is K*-bilipschitz* if

$$\forall x \neq x' \in X,\ \ 1/K \leq d_Y(fx, fx')/d_X(x, x') \leq K.$$

Two metric spaces are close in the Lipschitz sense if there is a $(1 + \epsilon)$-bilipschitz map between them with ϵ small.