Chapter 5

Deformations of Hyperbolic Structures

5.1 Introduction

Let Q be a compact 3-dimensional orbifold with link singularities. In most cases the manifold $Q - \Sigma$ obtained by removing the singular locus has a finite volume hyperbolic structure. If Q has a hyperbolic structure, it can be viewed as a cone-manifold structure with cone angles of the form $2\pi/m$. In the proof of the Orbifold Theorem we will attempt to connect the complete structure on $Q-\Sigma$, viewed as a cone-manifold with angles 0, with the desired orbifold structure via a family of cone-manifolds.

To study hyperbolic cone-manifold structures on Q with singularities along Σ, we first remove a neighbourhood of Σ from Q to obtain a compact manifold M with boundary consisting of tori. First we investigate when deformations of a hyperbolic structure on M exist. We will show that hyperbolic (or general (G, X)) structures on a compact manifold M are locally in 1–1 correspondence with nearby holonomy representations $\pi_1(M) \to G$ up to conjugacy. (If M has boundary, we may have to restrict to the complement of a small neighbourhood of the boundary ∂M.)

We then study how the deformed hyperbolic structures behave near the boundary of M. We will see that to find a nearby cone-manifold structure, it suffices to find a nearby holonomy representation for which the holonomy of each meridian is elliptic.

Next we discuss Thurston's analysis of representation spaces for 3-manifold groups into $PSL(2, \mathbb{C})$ and his theory of hyperbolic Dehn surgery. In particular, this implies that hyperbolic cone-manifold structures on Q with cone