Chapter 2

Orbifolds

An *orbifold* is a space locally modelled on the quotients of Euclidean space by finite groups. Further, these local models are glued together by maps compatible with the finite group actions. Natural examples are obtained from the quotient of a manifold by a finite group, but not all orbifolds arise in this way.

Given any idea in 3-dimensional topology, one can try to quotient out (locally) by finite group actions. This leads to orbifold versions of concepts including: covering space, fundamental group, submanifold, incompressible surface, prime decomposition, torus decomposition, bundle, Seifert fibre space, geometric structure. This chapter will review some of these basic concepts for orbifolds, and state the main result in terms of orbifolds. See Thurston [84], Scott [73] and Bonahon-Siebenmann [11] for more details.

2.1 Orbifold definitions

Formally, a (smooth) orbifold consists of local models glued together with orbifold maps. A *local model* is a pair $(\tilde{U}, G)$ where $\tilde{U}$ is an open subset of $\mathbb{R}^n$ and G is a finite group of diffeomorphisms of $\tilde{U}$. We will abuse this terminology by saying the quotient space $U = \tilde{U}/G$ is the local model. An *orbifold map* between local models is a pair $(\tilde{\psi}, \gamma)$ where $\tilde{\psi} : \tilde{U} \longrightarrow \tilde{U}'$ is smooth, $\gamma : G \longrightarrow G'$ is a group homomorphism and $\tilde{\psi}$ is *equivariant*, that is $\tilde{\psi}(g\tilde{x}) = \gamma(g)\tilde{\psi}(\tilde{x})$ for all $g \in G$ and $\tilde{x} \in \tilde{U}$. Then $\tilde{\psi}$ induces a map $\psi : \tilde{U}/G \longrightarrow \tilde{U}'/G'$ and we will abuse terminology by saying it is an orbifold map. If γ is a monomorphism and $\tilde{\psi}, \psi$ are both injective we say that ψ is an *orbifold local isomorphism.*

An *n-dimensional orbifold* Q consists of a pair $(X_Q, \mathcal{U})$ where X_Q is the