Part IV
Appendix

A Vertex superalgebras

A.1 $\mathbb{Z}_2$-graded vector spaces

A vector space M with a direct sum decomposition $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is called a $\mathbb{Z}_2$-graded vector space. Elements of $M_{\bar{0}}$ are called even whereas those of $M_{\bar{1}}$ odd. We set

$$p(v) = \begin{cases} 0, & \text{if } v \text{ is even,} \\ 1, & \text{if } v \text{ is odd.} \end{cases}$$

For any $v \in M$, let v' (resp. v'') be the even (resp. odd) part of v: $v = v' + v''$ where $p(v') = 0$ and $p(v'') = 1$. We will abbreviate $(-1)^{p(v)}$ by $(-1)^v$, $(-1)^{p(u)p(v)}$ by $(-1)^{uv}$ and so on.

For a $\mathbb{Z}_2$-graded vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$, the space $\operatorname{End} M$ is canonically $\mathbb{Z}_2$-graded by setting

$$(\operatorname{End} M)_{\bar{i}} = \{a \in \operatorname{End} M \mid a(M_{\bar{j}}) \subset M_{\bar{i}+\bar{j}}\},$$

namely, by setting $a'(v) = a(v')' + a(v'')''$ and $a''(v) = a(v')'' + a(v'')'$.

The supercommutator of $a, b \in \operatorname{End} M$ is defined by

$$[a, b] = (a'b' - b'a') + (a'b'' - b''a') + (a''b' - b'a'') + (a''b'' + b''a'').$$

We will simply write this as $[a, b] = ab - (-1)^{ab}ba$. Then we have $[b, a] = -(-1)^{ab}[a, b]$ and the Jacobi identity

$$[[a, b], c] = [a, [b, c]] - (-1)^{ab}[b, [a, c]],$$

which is also written as

$$(-1)^{ca}[a, [b, c]] + (-1)^{ab}[b, [c, a]] + (-1)^{bc}[c, [a, b]] = 0.$$

Let M and N be $\mathbb{Z}_2$-graded vector spaces. The tensor product $M \otimes N$ of vector spaces is canonically $\mathbb{Z}_2$-graded by setting

$$(M \otimes N)_{\bar{0}} = (M_{\bar{0}} \otimes N_{\bar{0}}) \oplus (M_{\bar{1}} \otimes N_{\bar{1}}), \quad (M \otimes N)_{\bar{1}} = (M_{\bar{0}} \otimes N_{\bar{1}}) \oplus (M_{\bar{1}} \otimes N_{\bar{0}}).$$

The tensor product of $a \in \operatorname{End} M$ and $b \in \operatorname{End} N$ is defined by

$$(a \otimes b)(u \otimes v) = (-1)^{bu} a(u) \otimes b(v).$$