Part III

Topics and examples

In Section 7, we will explain some particular classes of vertex algebras and describe the notion of an invariant bilinear form. Some relations to other algebraic objects such as Lie algebras, commutative algebras and associative algebras are described in Section 8. Section 9 is devoted to describing some famous examples: the vertex algebras associated to the free boson, to the β-γ system, to the affine Lie algebras, to the Virasoro algebra and to the $W_{1+\infty}$ algebra, and the lattice vertex algebra.

7 Summary of related notions

In this section, we will give a brief survey of various notions such as gradings, quasiconformal structures, conformal structures and invariant bilinear forms. We also summarize the notion and terminologies on vertex operator algebras.

7.1 Gradings of a vertex algebra

We first review the notion of a grading.

Definition 7.1.1. A *grading* of a vertex algebra V is a direct sum decomposition $V = \oplus_r V^r$, where r runs over a set of scalars, such that

$$V^r_{(n)} V^s \subset V^{r+s-n-1} \tag{7.1.1}$$

for all $n \in \mathbb{Z}$. A graded vertex algebra is a vertex algebra equipped with a grading.

A vector of a graded vertex algebra V is said to be homogeneous of degree r if it belongs to V^r. We denote the degree of a homogeneous vector a by $\Delta(a)$. (Whenever mentioning $\Delta(a)$, the vector a is implicitly supposed to be homogeneous).

For a graded vertex algebra V, we consider the operator $D : V \longrightarrow V, Da = \Delta(a)a$. Then the condition (7.1.1) is written as

$$D(a_{(n)}b) = (Da)_{(n)}b + a_{(n)}(Db) - (n+1)a_{(n)}b \tag{7.1.2}$$

which is equivalent to

$$[D, Y(a,z)] = z\partial Y(a,z) + \Delta(a)Y(a,z).$$