the following bijection,

$$\{\text{link diagrams with decompositions}\}/\text{moves of types 1, 2 and 3}$$
$$= \{\text{links}\}/\text{isotopy}.$$

Outline of the proof. We show outline of the proof in the following three steps.

Step1. We show

$$\begin{aligned}&\{\text{link diagrams with decompositions}\}/\text{moves of type1}\\ &= \{\text{link diagrams}\}/\text{restricted isotopy of } \mathbb{R}^2,\end{aligned} \tag{1.12}$$

where the isotopy in the right hand side is restricted such that the height function of $\sqcup^l S^1$,

$$\sqcup^l S^1 \to \mathbb{R}^2 \to \mathbb{R},$$

is preserved. Here the first map is an immersion of $\sqcup^l S^1$ expressing a link diagram, and the second map is the projection to the height coordinate. The formula (1.12) is reduced to the following formula,

$$\{\text{the trivial quasi-tangles with decompositions}\}/\text{moves of type1}$$
$$= \{\text{the trivial quasi-tangles}\},$$

which is shown by elementary calculations.

Step2. The following formula holds

$$\begin{aligned}&\{\text{link diagrams with decompositions}\}/\text{moves of types 1 and 2}\\ &= \{\text{link diagrams}\}/\text{isotopy of } \mathbb{R}^2.\end{aligned} \tag{1.13}$$

This equality is shown by (1.12) in Step 1 and results in [40].

Step 3. We obtain the required formula by (1.13) in Step 2 and Reidemeister's theorem, see [5]. □

2 The modified Kontsevich invariant

2.1 Definition of the modified Kontsevich invariant

Let L be an oriented framed link with l components in S^3. We fix a link diagram of L such that the framing of L is expressed by the blackboard framing of the link