0 Introduction

Discovery of the Jones polynomial[1] [10] was the beginning of the series of discoveries of new many invariants of knots (and links), known as quantum invariants at the present. After the Jones polynomial, Turaev [39] defined link invariants derived from solutions of the Yang-Baxter equation obtained from representations of quantum groups, and Kirillov-Reshetikhin [14] constructed link invariants directly by using representations of the quantum group $U_q(sl_2)$. Further it became known that, for each Lie algebra $\mathfrak{g}$ and each representation R of it, there exists an invariant of links derived from the quantum group $U_q(\mathfrak{g})$ of $\mathfrak{g}$ and the representation of the quantum group obtained by deforming R. We call the invariant the *quantum* $(\mathfrak{g}, R)$ *invariant* of links.

In 1989, Witten proposed his famous formula in [42], written by using a path integral based on Chern-Simons gauge theory. The integral is over the infinite dimensional set of G connections on a 3-manifold, where G is a fixed Lie group. In particular, when the 3-manifold is S^3 including a link, his formula expresses a quantum invariant of the link. Though nobody gives a geometric regularization of the formula yet, the formula gave us a new viewpoint to understand quantum invariants. Further the formula also gave a predict of existence of an invariant of 3-manifolds for each Lie group G. We call the invariant the *quantum* G *invariant* of 3-manifolds.

In a combinatorial viewpoint, also as predicted by Witten's formula, the quantum G invariant of a 3-manifold should be obtained as a linear sum of quantum invariants of a link in S^3 such that the 3-manifold is obtained by Dehn surgery along the link. Along the idea, Reshetikhin and Turaev [34] gave a rigorous construction of the quantum $SU(2)$ invariant of 3-manifolds. It was a new invariant of 3-manifolds obtained by a new construction, different from "classical invariants" such as homology groups or the fundamental group. Though the construction is not a natural geometric realization of Witten's idea,

[1] It is interpreted as the quantum (sl_2, V) invariant of knots later, where V is the vector representation of the Lie algebra sl_2.