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Billey's formula in combinatorics, geometry, and topology

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§1. Introduction

In this paper we describe a powerful combinatorial formula and its implications in geometry, topology, and algebra. This formula first appeared in the appendix of a book by Andersen, Jantzen, and Soergel [1, Appendix D]. Sara Billey discovered it independently five years later, and it played a prominent role in her work to evaluate certain polynomials closely related to Schubert polynomials [4].

To set the stage for our discussion, we review well-known foundations of Schubert calculus in Lie type A_{n-1}. Consider the group of invertible matrices $GL_n(\mathbb{C})$ with the subgroup B of upper-triangular matrices. The flag variety is the quotient $GL_n(\mathbb{C})/B$ and can be thought of geometrically as the collection of nested vector subspaces $V_1 \subseteq V_2 \subseteq \cdots V_{n-1} \subseteq \mathbb{C}^n$ where each V_i is i-dimensional. The geometry of the flag variety is interwoven with the combinatorics of the permutation group: the torus T of diagonal matrices in B acts on the flag variety, and its fixed points are the flags corresponding to permutation matrices. For each permutation w, the double coset BwB is an affine cell inside the flag variety, and the union of these double cosets forms a CW-decomposition. The closures of the cells $\overline{BwB/B}$ are the Schubert varieties, which induce a basis for the cohomology of the flag variety. Combinatorial properties

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