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On symplectic hypersurfaces

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§1. Introduction

A symplectic variety is a normal complex variety X with a holomorphic symplectic form ω on the regular part X_{reg} and with rational Gorenstein singularities. Affine symplectic varieties arise in many different ways such as closures of nilpotent orbits of a complex simple Lie algebra, as Slodowy slices to such nilpotent orbits or as symplectic reductions of holomorphic symplectic manifolds with Hamiltonian actions. Many examples of affine symplectic varieties tend to require large embedding codimensions compared to their dimensions.

In this article we treat the rarest case, namely affine symplectic *hypersurfaces*. For technical reasons we also impose the condition that X admit a good $\mathbb{C}^*$-action, i.e. that its affine coordinate ring $A = \mathbb{C}[X]$ be positively graded, $A = \oplus_{i \geq 0} A_i$ with $A_0 = \mathbb{C}$, and that ω is also homogeneous of positive weight s. This condition is satisfied in all examples we know. Finally, such a homogeneous symplectic hypersurface X is called *indecomposable* if the unique fixed point of the $\mathbb{C}^*$-action is a Poisson subscheme of X. As the term indecomposable indicates, such singularities are essential factors of more general hypersurfaces in the sense that every homogeneous hypersurface (X, ω) equivariantly decomposes into a product $W_1 \times ... \times W_k \times X'$, where X' is an indecomposable homogeneous hypersurface and each W_i is isomorphic to $\mathbb{C}^2$ with a standard symplectic form of the same weight s as ω (Lemma 2.5).

Indecomposable homogeneous hypersurfaces $X = \{f = 0\} \subset \mathbb{C}^{2n+1}$ have the remarkable property that the Poisson structure $\{-,-\} : A \times A \to A$ defined on the coordinate ring A by the symplectic structure

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